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Rotating n-gon/kn-gon Vortex Configurations

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Summary. We demonstrate the existence of stationary point-vortex configurations consisting of k vortex n-gons and a vortex kn-gon. These configurations exist only for specific values of the vortex strengths; the relative vortex strengths of such a configuration can be uniquely expressed as functions of the radii of the polygons. The kn-gon must be oriented so as to be fixed by any reflection fixing one of the n-gons; for sufficiently small k, we show that the n-gons must be oriented in such a way that the entire configuration shares the symmetries of any of the n-gons. Necessary conditions for the formal stability of general stationary point-vortex configurations set conditions on the vortex strengths. We apply these conditions to the n-gon/kn-gon configurations and carry out a complete linear and formal stability analysis in the case $k = n = 2$, showing that linearly and nonlinearly orbitally stable configurations exist.

1. Introduction

Much attention has been devoted to the study of *vortex polygons* or *rings*, i.e., vortex configurations in which some sets of vortices of equal strength are arranged so as to form the vertices of polygons. Most studies of vortex polygons have considered only collections of rings of equal numbers of vortices, that is, configurations consisting of k collections of n vortices each, with all n vortices in each collection being of equal strength. See Aref [1982], [1983], [1995], Campbell and Ziff [1978], [1979], or Koiller et al. [1985], and references therein for detailed discussions of such configurations. If the polygons are all n-gons for some integer n and all vortices in each n-gon have equal vortex strength, then if the vortices are initially in a polygonal configuration, they will remain in such a configuration for all time. The motion of each polygon is determined by the motion of a single vortex in the polygon.

We consider a situation which in some regards is an obvious generalization of the class of configurations considered above and yet differs dramatically in other aspects. If the configuration contains vortex polygons consisting of unequal numbers of vertices, then an initially polygonal configuration typically will not retain its polygonal character.

However, some initial conditions do preserve the polygonal structure for all time. In particular, there are families of polygonal configurations that evolve by a steady rotation.

We focus our attention on a family of point vortex configurations consisting of k n-gons and one kn-gon such that the vortex strengths of each of the polygons are distinct and the polygons have a common center. These configurations are in many regards more delicate than collections of n-gons alone. Such configurations exist only for very specific combinations of ring radii and vortex strengths. The relative orientations of the polygons are also highly restricted: The kn-gon must be fixed by reflections fixing any of the n-gons; for sufficiently small values of k we can show that, in addition, all of the n-gons must be fixed by reflections fixing any one of the n-gons. Our analysis emphasizes the role of symmetry in identifying the possible n-gon/kn-gon polygonal configurations.

We also study the stability of some rigidly rotating vortex configurations. Most of our discussion focuses on formal stability, i.e., definiteness modulo infinitesimal symmetries of the restriction of the second variation of an appropriate combination of the energy and the momentum of the system to the space of momentum-preserving variations. In this setting formal stability implies nonlinear orbital stability; the implications of indefiniteness are more subtle. We develop some necessary conditions for formal stability with respect to general variations for arbitrary rigidly rotating configurations and analyse in detail the formal and linear stability of configurations consisting of two vortex pairs and a vortex quartet. We again exploit the symmetries of the system, using them to construct a candidate Lyapunov function for the system and to streamline the analysis of that function.

2. Collections of Vortices of Unequal Strength

Point vortex models are highly idealized dynamical systems intended to capture some of the qualitative features of the motion of vortices in fluids. (See, for example, Aref [1983] for an overview of point vortex models.) We shall not address the physical interpretation of point vortex models here; rather, we shall emphasize the geometric structure of such systems. The equations of motion for N planar point vortices $z_1, \ldots, z_N \in \mathbb{C}$ with vortex strengths $\Gamma_1, \ldots, \Gamma_N$ are

$$\dot{\bar{z}}_\alpha = \frac{1}{2\pi i} \sum_{\beta=1,\beta\neq\alpha}^{N} \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \tag{1}$$

It was shown by Kirchhoff [1876] that the equations (1) are Hamiltonian, with Hamiltonian

$$H(z_1, \ldots, z_N) = -\frac{1}{8\pi} \sum_{\alpha,\beta=1,\alpha\neq\beta}^{N} \Gamma_\alpha \Gamma_\beta \log \left| z_\alpha - z_\beta \right|^2 \tag{2}$$

and constant noncanonical symplectic structure

$$\omega((v_1, \ldots, v_N), (w_1, \ldots, w_N)) = \sum_{\alpha=1}^{N} \Gamma_\alpha \mathrm{Im}\,[v_\alpha \bar{w}_\alpha]\,. \tag{3}$$

At present we need only the equations of motion (1). However, we shall make explicit use of this Hamiltonian structure in the stability analysis.

2.1. *k n-gons, One kn-gon Configurations*

We consider collections of vortices in which several of the vortices have equal strength. The situation in which a collection of kn vortices can be partitioned into k groups of n vortices of equal strength has been studied by several authors, including Aref [1982] and Koiller et al. [1985]. The special case of two groups of vortices with vortex strengths of equal magnitude and opposite sign is the classical Havelock [1931] problem. Much effort has been devoted to the study of *vortex polygons* or *rings*, i.e., sets of vortices of equal strength which are positioned on the vertices of polygons. We shall say that the vortices $z_{\iota_0}, \dots, z_{\iota_{n-1}}$ form a vortex n-gon if

$$\Gamma_{\iota_j} = \Gamma_{\iota_0} \qquad \text{and} \qquad \sum_{\ell=0,\ell\neq j}^{n-1} z_{\iota_\ell} = z e^{2\pi j/n}, \qquad j = 0, \dots, n-1, \tag{4}$$

for some $z \in \mathbb{C}$.

Here we consider partitions of unequal size; specifically, we study a collection of kn vortices of strength Γ and k sets of n vortices each of strength $\Gamma\,\gamma_i$, $i = 0, \dots, k-1$. The positions of the kn vortices of strength Γ are denoted z_α, $\alpha = 0, \dots, kn-1$, while those of strength $\Gamma\,\gamma_i$ are denoted $\zeta_{j\beta}$, $\beta = 0, \dots, n-1$, $\quad j = 0, \dots, k-1$. The equations of motion (1) for this collection of vortices take the form

$$\dot{\bar{z}}_\alpha = \frac{\Gamma}{2\pi i}\left(\sum_{\ell=0}^{kn-1} (z_\alpha - z_\ell)^{-1} + \sum_{m=0}^{k-1} \gamma_m \sum_{\ell=0}^{n-1} (z_\alpha - \zeta_{m\ell})^{-1}\right) \tag{5}$$

for $\alpha = 0, \dots, kn-1$ and

$$\begin{aligned}\dot{\bar{\zeta}}_{j\alpha} &= \frac{\Gamma}{2\pi i}\left(\sum_{\ell=0}^{kn-1} \left(\zeta_{j\alpha} - z_\ell\right)^{-1} + \gamma_j \sum_{\ell=0,\ell\neq j}^{n-1} \left(\zeta_{j\alpha} - \zeta_{j\ell}\right)^{-1}\right.\\ &\qquad \left. + \sum_{m=0,m\neq j}^{k-1} \gamma_m \sum_{\ell=0}^{n-1} \left(\zeta_{j\alpha} - \zeta_{m\ell}\right)^{-1}\right)\end{aligned} \tag{6}$$

for $\alpha = 0, \dots, n-1$, $j = 0, \dots, k-1$.

We shall search for configurations consisting of k vortex n-gons and one vortex kn-gon, all centered at the origin. Such an arrangement corresponds to the assumption that there are complex-valued functions $z, \zeta_0, \dots, \zeta_{k-1}$ such that

$$\begin{aligned} z_\alpha(t) &= \Phi(\alpha)\, z(t), & \alpha &= 0, \dots, kn-1, \\ \zeta_{j\beta}(t) &= \Phi(k\beta)\, \zeta_j(t), & \beta &= 0, \dots, n-1, \quad j = 0, \dots, k-1, \end{aligned} \tag{7}$$

where $\Phi(x) := \exp\left(\frac{2\pi i x}{kn}\right)$, for all times t. In this situation, there are $k+1$ polygons: one kn-gon with 'leader' z_0 and vortex strengths Γ, and k n-gons, each with 'leader' ζ_{j0} and vortex strengths $\Gamma\,\gamma_j$.

If (7) holds, then $z_\alpha \dot{\bar{z}}_\alpha = z\dot{\bar{z}}$ for $\alpha = 1, \ldots, kn-1$ and $\zeta_{j\beta}\dot{\bar{\zeta}}_{j\beta} = \zeta_j\dot{\bar{\zeta}}_j$ for $j = 0, \ldots, k-1$ and $\beta = 1, \ldots, n-1$. Thus, if we define $\rho_j := (\zeta_j/z)^n$ for $j = 0, \ldots, k-1$, then (5), (6), and (7) imply that

$$\begin{aligned}
\frac{2\pi i\, z\dot{\bar{z}}}{\Gamma} &= \frac{2\pi i\, z_\alpha \dot{\bar{z}}_\alpha}{\Gamma} \\
&= \sum_{\ell=0,\ell\neq\alpha}^{kn-1} (1 - \Phi(\ell-\alpha))^{-1} + \sum_{j=0}^{k-1} \gamma_j \sum_{\ell=0}^{n-1} \left(1 - \Phi(k\ell-\alpha)\,\zeta_j/z\right)^{-1} \\
&= \frac{kn-1}{2} + n \sum_{j=0}^{k-1} \frac{\gamma_j}{1 - \Phi(-n\,\alpha)\,\rho_j}
\end{aligned} \tag{8}$$

for $\alpha = 1, \ldots, kn-1$, and

$$\begin{aligned}
\frac{2\pi i\, \zeta_j\dot{\bar{\zeta}}_j}{\Gamma} &= \frac{2\pi i\, \zeta_{j\beta}\dot{\bar{\zeta}}_{j\beta}}{\Gamma} \\
&= \sum_{\ell=0}^{kn-1} \left(1 - \Phi(\ell-k\beta)\,z/\zeta_j\right)^{-1} + \gamma_j \sum_{\ell=0,\ell\neq j}^{n-1} (1 - \Phi(k(\ell-\beta)))^{-1} \\
&\quad + \sum_{m=0}^{k-1} \gamma_m \sum_{\ell=0}^{n-1} \left(1 - \Phi(k(\ell-\beta))\,\zeta_m/\zeta_j\right)^{-1} \\
&= \frac{kn}{1-\rho_j^{-k}} + \frac{\gamma_j(n-1)}{2} + n \sum_{m=0}^{k-1} \frac{\gamma_m}{1 - \rho_m/\rho_j}
\end{aligned} \tag{9}$$

for $j = 0, \ldots, k-1$ and $\beta = 1, \ldots, n-1$. The last line of each equation is valid only if $\ell \neq j$ implies that $\rho_\ell \neq \rho_j$.

Remark. Equations (9) closely resemble the more familiar equations

$$\frac{2\pi i}{n}\zeta_{j\alpha}\dot{\bar{\zeta}}_{j\alpha} = \frac{(n-1)\gamma_j}{2n} + \sum_{\ell=0,\ell\neq j}^{k-1} \frac{\gamma_\ell}{1-\rho_\ell/\rho_j}, \qquad j = 0, \ldots, k-1, \tag{10}$$

associated with a system of k vortex n-gons, such as have been studied by Havelock [1931], Aref [1982], Koiller et al. [1985], and others. In particular, the index β does not appear on the right-hand side of the equations; hence these equations are automatically consistent with ring-type motions. The equations (8) are of a significantly different character—the index α identifying the individual vortices within the polygon appears explicitly in the right-hand side of the equations; we shall see that these equations are consistent with ring motions only for specific values of the vortex strengths and ring radii. The presence of the kn-gon sets severe restrictions on the possible polygonal configurations.

The equations (8) and (9) form a system of nonlinear constraint equations for the ρ_j's, parametrized in part by the constant vortex strengths $\gamma_0, \ldots, \gamma_{k-1}$. However, neither (8) nor (9) appears to possess closed form solutions. Thus, in the interest of obtaining analytic information about the behavior of such systems, we shall reverse the natural sense of the equations (8) and regard them as equations for the vortex strengths. We now derive an alternative expression for (8) which specifies the relative vortex strengths $\gamma_0, \ldots, \gamma_{k-1}$ in terms of the quantities ρ_j determined by the relative positions of the ring leaders.

Proposition 1. *The equations (8) are equivalent to the equations*

$$\gamma_j = \frac{\sigma\left(1-\rho_j^k\right)}{\prod_{\ell=0,\ell\neq j}^{k-1}(1-\rho_j/\rho_\ell)}, \qquad j = 0, \ldots, k-1, \tag{11}$$

where

$$\sigma = \frac{2\pi i z\dot{\bar{z}}}{\Gamma n} + \frac{1}{2n} - \frac{k}{2}. \tag{12}$$

Proof. The proof makes use of some basic facts about Vandermonde matrices and a closely related class of matrices. These facts are presented in Lemma 1, which is stated and proved in the Appendix.

Equation (8) implies that $z_\alpha \dot{\bar{z}}_\alpha = z\dot{\bar{z}}$ if and only if

$$\begin{aligned} 0 &= \sum_{j=0}^{k-1} \gamma_j \left(\frac{1}{1-\rho_j} - \frac{1}{1-\rho_j \Phi(-n\alpha)}\right) \\ &= \sum_{j=0}^{k-1} \frac{\gamma_j}{1-\rho_j^k} \sum_{\ell=1}^{k-1} \rho_j^\ell \left(1-\Phi(-n\ell\alpha)\right). \end{aligned} \tag{13}$$

Thus the configuration retains its polygonal character if and only if (13) holds for $\alpha = 1, \ldots, k-1$. If we define the $(k-1)\times(k-1)$ matrix $\mathcal{M}$ with $\alpha\ell$-th entry $1-\Phi(-n\,\alpha\,\ell)$ and the $(k-1)\times k$ matrix $\mathcal{N}$ with ℓj-th entry ρ_j^ℓ, then equation (13) is equivalent to the condition that

$$\left(\frac{\gamma_0}{1-\rho_0^k}, \ldots, \frac{\gamma_{k-1}}{1-\rho_{k-1}^k}\right) \in \ker\left[\mathcal{M}\,\mathcal{N}\right]. \tag{14}$$

The matrix $\mathcal{M}$ is invertible, as can be shown by setting $x_j = \Phi(-nj)$ in Lemma 1.1; thus $\ker\left[\mathcal{M}\,\mathcal{N}\right] = \ker\left[\mathcal{N}\right]$. By prepending the row $(1, \ldots, 1)$ to the matrix $\mathcal{N}$ we obtain the Vandermonde matrix $\mathcal{V}(\rho_0, \ldots, \rho_{k-1})$. Lemma 1.2 implies that

$$\ker\left[\mathcal{N}\right] = \operatorname{span}\left[\mathcal{V}(\rho_0, \ldots, \rho_{k-1})^{-1}(1, 0, \ldots, 0)\right] = \operatorname{span}\left[(\varphi_0, \ldots, \varphi_{k-1})\right], \tag{15}$$

where

$$\varphi_j := \frac{1}{\prod_{\ell=0,\ell\neq j}^{k-1}\left(1-\frac{\rho_j}{\rho_\ell}\right)}. \tag{16}$$

We note that $\mathcal{V}(x_1, \ldots, x_N)y = (1, 0, \ldots, 0)$ implies that

$$\sum_{j=1}^{N} \frac{y_j\left(1-x_j^N\right)}{1-x_j} = \sum_{j=1}^{N}\left(y_j \sum_{\ell=0}^{N-1} x_j^\ell\right) = \sum_{\ell=0}^{N-1}\left(\mathcal{V}(x_1, \ldots, x_N)y\right)_\ell = 1. \tag{17}$$

Thus $\gamma_0, \ldots, \gamma_{k-1}$ given by (11) satisfy

$$\sum_{j=1}^{N} \frac{\gamma_j}{1-\rho_j} = \sigma. \tag{18}$$

Comparing (8) and (18), we see that (8) holds if and only if (11) and (12) hold. □

We shall interpret (11) as determining the relative vortex strengths $\gamma_0, \ldots, \gamma_{k-1}$ as functions of $z\dot{\bar{z}}$, $\rho_0, \ldots, \rho_{k-1}$, and the parameters Γ, k, and n. We will then substitute the results into (9) and search for solutions of the resulting equations which yield constant γ_j's. In general, if the radii of the rings change, the relationships (11) between the radii and the vortex strengths will fail and the individual vortices will evolve in such a way that the polygonal structure is destroyed. Rather than search for the most general class of motions that 'preserve' the vortex strengths, we shall consider only rigidly rotating configurations, which yield constant values of the individual components $z\dot{\bar{z}}$, $\rho_0, \ldots, \rho_{k-1}$ of the right-hand sides of equations (11). Recent studies of stationary vortex configurations using geometric methods have been carried out by Palmore [1982], Koiller et al. [1985], and Kirwan [1988], among others.

2.2. Rotating Polygonal Configurations

A kn-gon/n-gons configuration rotates steadily if there exists $\xi \in \mathbb{R}$ such that $z_\alpha(t) = e^{i\xi t} z_\alpha(0)$ for $\alpha = 0, \ldots, kn-1$ and $\zeta_{j\beta}(t) = e^{i\xi t}\zeta_{j\beta}(0)$ for $j = 0, \ldots, k-1$ and $\beta = 0, \ldots, n-1$. Comparing this condition with equations (9), (11), and (12), we see that the ring configuration determined by (7) evolves by a steady rotation with angular velocity ξ if and only if

$$\sigma = \frac{2\pi\xi\, |z|^2}{n\,\Gamma} + \frac{1}{2n} - \frac{k}{2} \tag{19}$$

and

$$\left(\sigma - \frac{1}{2n} + \frac{k}{2}\right) |\rho_j|^{2/n} = \frac{k}{1-\rho_j^{-k}} + \frac{\gamma_j(n-1)}{2n} + \sum_{\ell=0, \ell\neq j}^{k-1} \frac{\gamma_j}{1-\rho_\ell/\rho_j},$$

$$j = 0, \ldots, k-1, \tag{20}$$

where the γ_j's are given by (11). The assumption that the vortices all rotate at equal velocity about the origin implies that $z\dot{\bar{z}}$ and the ρ_j's are constant. Thus the condition that the expressions (11) for the relative vortex strengths $\gamma_0, \ldots, \gamma_{k-1}$ be constant is automatically satisfied. The remaining $k-1$ equations (20) can be solved for the ρ_j's.

The $O(2)$ symmetry of the system (1) imposes the condition on rigidly rotating relative equilibria that the lines passing from the origin to the vertices of the n-gons either coincide with one of the lines from the origin to a vertex of the kn-gon (a 'vertex-vertex' orientation) or lie precisely halfway between two such lines (a 'vertex-face' orientation). See Figure 1 for examples of such configurations. This condition implies

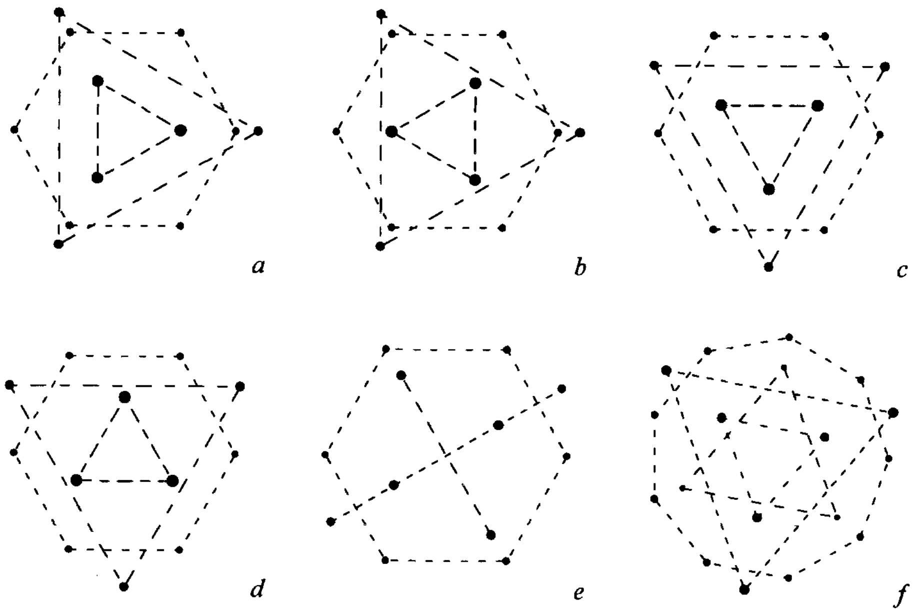

Fig. 1. Representative admissible orientations. (a) $(k, n) = (2, 3)$, m and $\phi_0 - \phi_1$ even; (b) $(k, n) = (2, 3)$, m even, $\phi_0 - \phi_1$ odd; (c) $(k, n) = (2, 3)$, m odd, $\phi_0 - \phi_1$ even; (d) $(k, n) = (2, 3)$, m and $\phi_0 - \phi_1$ odd; (e) $(k, n) = (3, 2)$, $\phi_0 - \phi_1$ even, $\phi_0 - \phi_2$ odd; (f) $(k, n) = (3, 3)$, $\phi_0 - \phi_1$ even, $\phi_0 - \phi_2$ odd.

that the kn-gon is fixed by the reflections across the lines through the vertices of any of the n-gons. An additional condition on the ρ_j's follows from this symmetry condition. Specifically:

Proposition 2. *If $\zeta_0, \ldots, \zeta_{k-1}$ determine a relative equilibrium in steady rotation about the origin, then*

1. $\zeta_j/z = r_j \Phi\left(m_j/2\right)$, *where $r_j \in \mathbb{R}^+$ and $m_j \in \{0, \ldots, 2k-1\}$, for $j = 0, \ldots, k-1$. Equivalently, $\rho_j^k \in \mathbb{R}$ for $j = 0, \ldots, k-1$.*
2. $\prod_{\ell=0,\ell\neq j}^{k-1}(1 - \rho_j/\rho_\ell) \in \mathbb{R}$ *for $j = 0, \ldots, k-1$.*

Proof. The equations of motion (1) are equivariant with respect to the action of $O(2)$ if orientation-reversing transformations are accompanied by a time reversal. Reflections across the line passing through the origin and a point z_* map z to $\bar{z}\, z_*/\bar{z}_*$. Thus, if we reflect across the line through ζ_j, then ρ_j/ρ_ℓ is mapped to $\overline{(\rho_j/\rho_\ell)}$. Consider equation (11) for the relative vortex strengths; since a relative equilibrium must remain a relative equilibrium after reflection, the equation for the γ_j's must be fixed by reflection. The component $1 - \rho_j^k$ for γ_j is unchanged by a reflection fixing ζ_j, so we must have

$$\prod_{\ell=0,\ell\neq j}^{k-1} (1 - \rho_\ell/\rho_j) = \prod_{\ell=0,\ell\neq \ell}^{k-1} \left(1 - \overline{(\rho_\ell/\rho_j)}\right) = \overline{\prod_{\ell=0,\ell\neq j}^{k-1} (1 - \rho_\ell/\rho_j)}. \tag{21}$$

Thus the second condition holds. Since $\gamma_j \in \mathbb{R}$, it follows from (11) that $\rho_j^k \in \mathbb{R}$ as well. □

The second condition of Proposition 2 is clearly satisfied if $\rho_\ell/\rho_j \in \mathbb{R}$ for all j and ℓ. For small k it is relatively easy to show that these ratios must be real.

Corollary 1. *If $\zeta_0, \ldots, \zeta_{k-1}$ determine a relative equilibrium in steady rotation about the origin for $k = 2, 3$ or 4, then $\rho_\ell/\rho_j \in \mathbb{R}$ for $j = 0, \ldots, k-1$.*

Proof. For $k = 2$, the result follows immediately from the second condition of Proposition 2.

Given $\zeta_0, \ldots, \zeta_{k-1}$, let r_j and m_j, $j = 0, \ldots, k-1$, be given as in the first condition of Proposition 2. For notational convenience, set $R_j := r_j^n$ and define $s_k\colon \mathbb{Z} \to [-1, 1]$ by $s_k(m) := \sin(\pi m/k)$. Regrouping terms, we see that if $k = 3$, then the second condition holds for ρ_0, ρ_1, ρ_2 if and only if the vector (R_0, R_1, R_2) lies in the kernel of the matrix

$$\begin{pmatrix} s_k(2m_0 - m_1 - m_2) & s_k(m_2 - m_0) & s_k(m_1 - m_0) \\ s_k(m_2 - m_1) & s_k(2m_1 - m_0 - m_2) & s_k(m_0 - m_1) \\ s_k(m_1 - m_2) & s_k(m_0 - m_2) & s_k(2m_2 - m_0 - m_1) \end{pmatrix}. \tag{22}$$

A simple calculation shows that the kernel of the matrix (22) is one-dimensional unless the m_j's are all equal. These kernels are spanned by vectors whose entries are permutations of the entries of one of the following three vectors: $(1, -1, 1)$, $(1, 2, 1)$, or $(1, -2, 1)$. Thus either the m_j's are all equal, which implies that the ratios ρ_ℓ/ρ_j are all real, or else two of the R_j's are equal, which violates the condition that the radii of the polygons are distinct.

The argument for $k = 4$ is completely analogous, but computationally more tedious. The product $\prod_{j=1}^{3}(1 - \rho_0/\rho_j)$ is real if and only if

$$\begin{aligned} &R_0^2 s_k(3m_0 - m_1 - m_2 - m_3) \\ &\quad - R_0\,(R_1\, s_k(2m_0 - m_2 - m_3) + R_2\, s_k(2m_0 - m_1 - m_3) + R_3\, s_k(2m_0 - m_1 - m_2)) \\ &\quad + R_1 R_2\, s_k(m_0 - m_3) + R_1 R_3\, s_k(m_0 - m_2) + R_2 R_3\, s_k(m_0 - m_1) = 0. \end{aligned} \tag{23}$$

The conditions for the reality of the other three products are obtained by cyclically permuting the indices in (23). Thus we obtain a collection of systems of four quadratic equations in the four unknowns $R_0, \ldots, R_3$ indexed by the m_j's. These equations can be solved numerically; for all of the solutions for which the m_j's are not all equal, either $|R_j| = |R_\ell|$ for some $j \neq \ell$ or $|R_j| = 1$ for some j, both of which imply that two of the polygons have equal radii. Thus the only possible relative equilibria for $k = 4$ are those for which $\rho_j/\rho_\ell \in \mathbb{R}$ for all j and ℓ. □

Remark. In general, the condition that the products be real valued is equivalent to a system of k equations, each equation being a $(k-2)$-th order polynomial in the R_j's with coefficients determined by the angles m_j. Clearly some argument other than explicit solution of the equations is required to show for arbitrarily large n that the n-gons must have a common symmetry group.

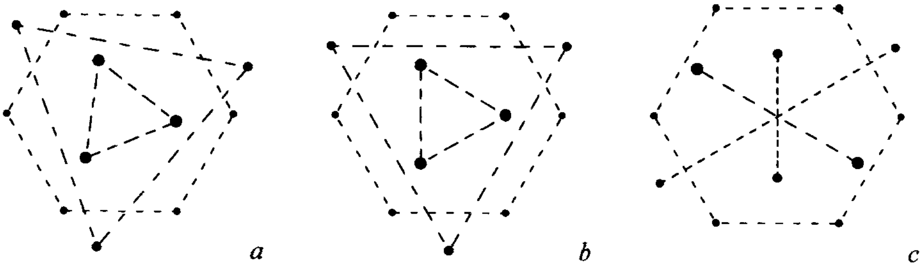

Fig. 2. Sample inadmissible orientations: (a) $(k, n) = (2, 3)$, $\arg \zeta_0 = \frac{\pi}{9}$, $\arg \zeta_1 = \frac{5\pi}{8}$; (b) $(k, n) = (2, 3)$, $\arg \zeta_0 = 0$, $\arg \zeta_1 = \frac{\pi}{6}$; (c) $(k, n) = (3, 2)$, $\arg \zeta_0 = \frac{\pi}{6}$, $\arg \zeta_1 = \frac{\pi}{2}$, $\arg \zeta_2 = \frac{5\pi}{6}$.

The ratios ρ_ℓ/ρ_j are all real valued if and only if $\zeta_j/z = r_j \Phi\big((k\phi_j + m)/2\big)$ for $r_j \in \mathbb{R}^+$ and integers $\phi_0, \ldots, \phi_{k-1}$ and m. These are configurations of maximal symmetry in the following sense. If we do not label the vortices, but say that a configuration is fixed by a transformation if it takes each vortex to another vortex of equal strength, then by construction all of the n-gon/kn-gon configurations considered here are fixed by the group D^n of rotations through the angle $2\pi/n$. As was discussed before, the necessary condition $\rho_\ell^k \in \mathbb{R}$ implies that the kn-gon is fixed by reflections across lines through any of the vertices of the n-gons. If $\rho_\ell/\rho_j \in \mathbb{R}$ for all j and ℓ, then all of the n-gons are also fixed by these reflections.

The analysis of the possible symmetries of the relative equilibria not only provides important qualitative information, but also greatly simplifies the calculation of specific quantitative information. Proposition 2 implies that the equilibrium conditions (9) simplify to a system of k rational equations

$$\left(\sigma + \frac{k}{2} - \frac{1}{2n}\right) r_j^2 = \frac{k}{1 - (-1)^{m+k\phi_j} r_j^{-kn}} + \sigma \left(\frac{(n-1)\tilde{\gamma}_j}{2n} + \sum_{\ell=0, \ell \neq j}^{k-1} \frac{\tilde{\gamma}_\ell}{1 - (-1)^{(\phi_\ell - \phi_j)} (r_\ell/r_j)^n} \right) \tag{24}$$

in the real variables r_j, where

$$\tilde{\gamma}_j := \frac{1 - (-1)^{m+k\phi_j} r_j^{kn}}{\prod_{\ell=0,\ell\neq j}^{k-1} \left(1 - (-1)^{(\phi_j - \phi_\ell)} (r_j/r_\ell)^n\right)}. \tag{25}$$

The system of equations (19) and (24) is relatively easy to solve numerically in comparison to the general system of equations (20).

Figure 1 illustrates some representative admissible n-gon orientations.

Figure 2 illustrates some *inadmissible* orientations: Figure 2.a shows a 'random' orientation of two triangles and a hexagon ($k = 2$, $n = 3$) which violates the first equilibrium condition given in Proposition 2; reflections preserving the triangles do not fix the hexagon. Figures 2.b and 2.c violate the condition that the ratios ρ_j/ρ_ℓ be real for $k = 2, 3, 4$; note that in these two examples reflections preserving one of the n-gons fix the hexagon, but fail to preserve the other n-gons.

Remark. The equations for rigidly rotating relative equilibria with generator ξ corresponding to the equations (10) for k n-gons are

$$2\pi\xi\, |\rho_j|^{2/n} = \frac{(n-1)\gamma_j}{2n} + \sum_{\ell=0,\ell\neq j}^{k-1} \frac{\gamma_\ell}{1-\rho_\ell/\rho_j}, \qquad j = 0, \ldots, k-1. \tag{26}$$

As before, we can reverse the natural interpretation of the equations and view (26) as a system of k linear equations in the 'unknowns' $\gamma_0, \ldots, \gamma_{k-1}$ parametrized by the real parameter ξ and the complex k vector $\boldsymbol{\rho} = (\rho_0, \ldots, \rho_{k-1})$. Following Koiller et al. [1985], we define the complex $k \times k$ matrix $\mathbf{A}(\boldsymbol{\rho})$ and the real k vector $\mathbf{b}(\boldsymbol{\rho})$ with entries

$$a_{j\ell} = \begin{cases} \frac{n-1}{2n}, & j = \ell \\ \frac{1}{1-\rho_j/\rho_\ell}, & j \neq \ell \end{cases} \qquad \text{and} \qquad b_j = 2\pi\xi\, |\rho_j|^{2/n}. \tag{27}$$

The vector $\boldsymbol{\rho}$ is a solution of (26) for some collection of vortex strengths $\gamma_0, \ldots, \gamma_{k-1}$ if and only if the equation $\mathbf{A}(\boldsymbol{\rho})\boldsymbol{x} = \mathbf{b}(\boldsymbol{\rho})$ has a real solution $\boldsymbol{\gamma}$. This condition sets restrictions on the set of possible $\boldsymbol{\rho}$'s; we can limit our search for relative equilibria to this set. For example, in the case $k = 2$ it is obvious that rotating relative equilibria exist only if $\rho_0/\rho_1 \in \mathbb{R}$, i.e., if the symmetry groups of the two vortex n-gons agree. Thus the vertices of one n-gon are either aligned with the vertices of the other or lie midway between them. (This fact is derived in Koiller et al. [1985] by means of an appropriate variation on the reduced phase space.)

2.3. Numerical Results

We now study numerically some of the solutions of (19) for $k = 2$. Corollary 1 states that in this case only configurations for which $\rho_0/\rho_1 \in \mathbb{R}$ can be relative equilibria. Thus there are only four possible orientations for the two vortex n-gons relative to the vortex $2n$-gon: the lines through the vortices in each n-gon intersect either the vertices or the midpoints of the faces of the other n-gon and the $2n$-gon. (See Fig. 1.) By testing each possible orientation separately, we can solve for the distance from the vortices to the origin (two real parameters), rather than the position of the vortices in each n-gon (two complex parameters).

We now introduce a change of variables that simplifies our search for solutions of (24). If we define $p := r_0 r_1$, $q := r_0/r_1$, $\phi := \phi_0 - \phi_1$,

$$P := (-1)^{m+\phi} p^n, \quad \text{and} \quad Q := (-1)^\phi q^{-n}, \tag{28}$$

then the equation (24) for $j = 0$ is equivalent to

$$\left(\sigma + 1 - \frac{1}{2n}\right) pq = \frac{2}{1 - Q/P} + \sigma\left(\frac{(n-1)(P-Q)}{2n(1-Q)} + \frac{1-PQ}{(1-Q)^2}\right). \tag{29}$$

Solving (29) for σ, we obtain

$$\sigma = f(p, q, n, m, \phi) := \frac{\dfrac{2}{Q/P - 1} + \left(1 - \dfrac{1}{2n}\right) pq}{\dfrac{(n-1)(P-Q)}{2n(1-Q)} + \dfrac{1-PQ}{(1-Q)^2} - pq}, \tag{30}$$

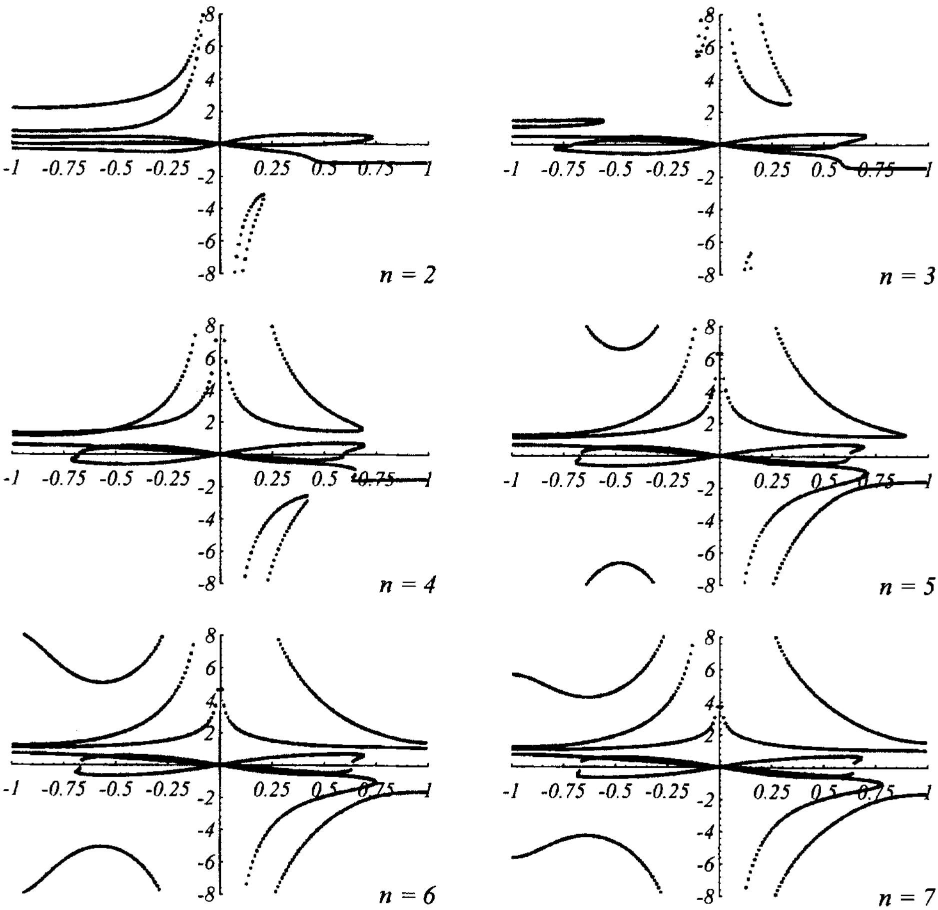

Fig. 3. Plots of relative equilibria for $k = 2, n = 2, \dots, 7$. Horizontal axis: $(-1)^{\phi}q$; vertical axis: $(-1)^{m+\phi}p$.

where P and Q are given by (28). The equation (24) for $j = 1$ is obtained by replacing q by $1/q$ in (29). Thus (29) is satisfied for both $j = 0$ and $j = 1$ only if $f(p, q, n, m, \phi) = f(p, 1/q, n, m, \phi)$. We solve this equation numerically for p, regarding the remaining variables as parameters. Note that the change of variables $(r_0, r_1) \to (\sqrt{pq}, \sqrt{p/q})$ has the computational advantage that we need only consider values of q in the finite range $0 < q < 1$. Solutions satisfying $p = q$ or $pq = 1$ must be discarded, since they violate the condition that the radii of the rings be distinct.

The results for some sample values of n are given in Figure 3. We have parametrized the problem in such a way that q and p are always positive, but we find it instructive to group the solution branches into a single plot of the pairs $\big((-1)^{\phi}q, (-1)^{m+\phi}p\big)$. We need only consider the values zero and one for m and ϕ, since only the parities of these parameters affect the equations. In the case $k = 2$ even values of m lead to 'vertex-vertex' configurations, as in Figures 1.a and 1.b, while odd values lead to 'vertex-face'

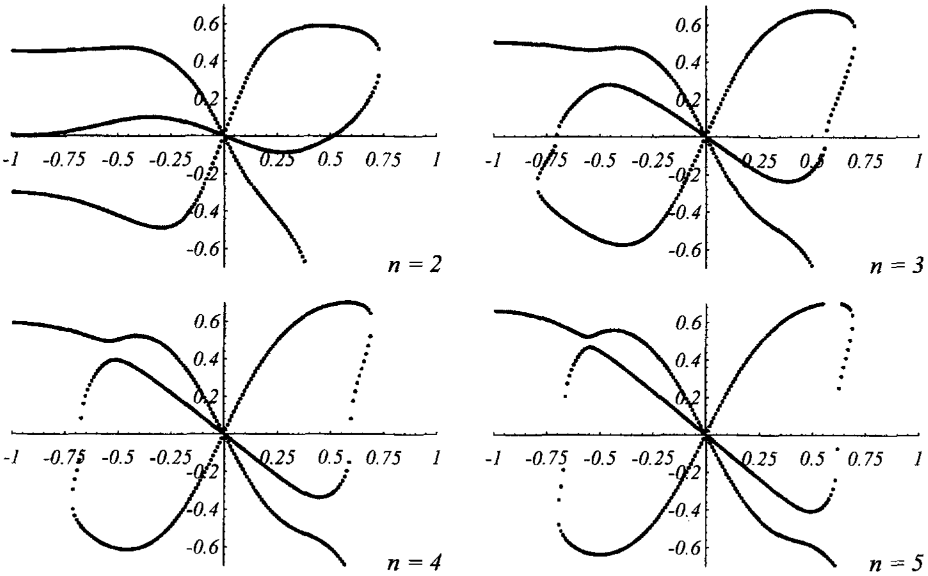

Fig. 4. Detail of 'bows' for $k = 2, n = 2, \ldots, 5$. Horizontal axis: $(-1)^{\phi}q$; vertical axis: $(-1)^{m+\phi}p$.

configurations, as in Figures 1.c and 1.d. Figure 4 shows the 'bow' equilibrium curves ($p < 3/4$) in detail for $n = 2, \ldots, 5$; note that the bow appears to approach a limiting curve as n increases. Plots of the relative vortex strengths γ_j and the angular velocity ξ for $k = 2, 3, 4$ are given in the Appendix.

3. Stability

The linear stability of point vortex configurations, particularly collections of vortices of equal strength, and related systems has been studied for over a century. See, e.g., Thomson [1883], Havelock [1931], Calogero [1978], Campbell [1981], and Aref [1995] for representative analyses. Nonlinear orbital stability analyses appear to be less common. The (non)existence of global Lyapunov functions for the von Kármán vortex model has been studied by Kochin et al. [1964] and Lim [1993]. A variational analysis of the nonlinear stability of certain steady motions of collections of vortex polygons with respect to perturbations preserving the polygonal structure, i.e., perturbations changing the radii and centers of the rings while maintaining the relative position of the vortices on each of the rings, has been carried out by Koiller et al. [1985]. For the vortex polygons discussed in Section 2, perturbation of the ring radii typically leads to motions which break the ring structure; hence a stability analysis within the class of ring motions does not seem to be particularly meaningful in the present setting. We consider the stability of the steady rotations of the vortex rings within the class of all possible motions for that number of vortices of the specified vortex strengths.

We first consider the nonlinear and linear stability of general point-vortex configurations and then apply our results to the polygonal vortex configurations discussed in the previous sections. Much of our discussion focuses on formal stability—the condition that an appropriate function have a semidefinite second variation, with a kernel consisting of infinitesimal group motions, at the relative equilibrium. Formal stability is closely related to nonlinear orbital stability. We shall see that a formally stable rotating relative equilibrium of the planar point-vortex system is nonlinearly stable modulo rotations about the origin. While the linear algebra required to determine the formal stability of configurations with more than a few vortices appears to typically be too complicated to be carried out analytically, there are readily identifiable classes of configurations which can easily be shown to be formally unstable. We shall fully determine the stability of only the smallest of the vortex polygon configurations discussed in Section 2—two pairs of vortices, each pair having equal vortex strength, and a quartet of vortices of equal strength.

To analyse the stability of the vortex rings, we make use of the Hamiltonian structure of the system. This structure plays a crucial role in the nonlinear stability analysis and is relevant to the linear stability analysis as well. Relative equilibria are critical points of an appropriate function, called the energy-momentum function, determined by the Hamiltonian and the conserved quantity associated with the rotation invariance of the system. Our motivation for this characterization is the following: If z is a local extremum of the energy-momentum function modulo rotations, then nearby level sets must be closed; hence trajectories which start sufficiently near the equilibrium orbit must remain near it for all time. In other words, the energy-momentum function serves as a Lyapunov function for the relative equilibrium z modulo rotations.

We first consider a general collection of N point vortices $z_1, \ldots, z_N$ and then apply our results to the polygonal configurations discussed in the previous sections. The Hamiltonian vector field of point vortex motion is determined by the differential of the Hamiltonian (2) through the relationship $DH(z)\cdot \boldsymbol{v} = \omega(X_H(z), \boldsymbol{v})$ for all $\boldsymbol{v} \in \mathbb{C}^N$, where ω denotes the symplectic structure (3). The configuration $z := (z_1, \ldots, z_N) \in \mathbb{C}^N$ is in steady rotation with angular velocity ξ if the Hamiltonian vector field X_H satisfies $X_H(z) = i\xi\, z$. As we shall show, the vector field $(\xi, \tau)_P(z) = i\xi\, z + \tau\, \boldsymbol{t}$, where $\xi \in \mathbb{R}$, $\tau \in \mathbb{C}$ and $\boldsymbol{t} := (1, \ldots, 1)$, is the Hamiltonian vector field of a function $J_{(\xi,\tau)}$. Thus, since the symplectic structure is nondegenerate, $X_H(z) = i\xi\, z$ if and only if z is a critical point of $H_\xi := H - J_{(\xi,0)}$. The stability analysis outlined above is an example of a general technique known as the energy-momentum method.

The *momentum map* collects the Hamiltonian vector fields for the infinitesimal symmetry group actions into a single object. We denote by $\boldsymbol{J}$ the momentum map $\boldsymbol{J}\colon \mathbb{C}^N \to \mathbb{R} \times \mathbb{C}$, given by

$$\boldsymbol{J}(z) = -\sum_{\alpha=1}^{N} \Gamma_\alpha \left(\tfrac{1}{2}\, |z_\alpha|^2\, ,\, i z_\alpha\right). \tag{31}$$

The momentum map $\boldsymbol{J}$ is determined by the following relations: For any $\xi \in \mathbb{R}$ and $\tau \in \mathbb{C}$, the vector field $(\xi, \tau)_P$ defined above consists of an infinitesimal rotation with angular velocity ξ and an infinite translation with velocity τ. The map $\boldsymbol{J}$ is a momentum map if and only if for any choice of ξ and τ the real-valued function $J_{(\xi,\tau)}(z) := \boldsymbol{J}(z)\cdot(\xi, \tau)$ has the Hamitonian vector field $X_{J_{(\xi,\tau)}} = (\xi, \tau)_P$, i.e., $DJ_{(\xi,\tau)} = \iota_{(\xi,\tau)_P}\omega$. (See Abraham and Marsden [1978] for a general discussion of momentum maps and energy methods; Adams and Ratiu [1988] discuss in detail the momentum map for point vortex systems.)

The straightforward calculation

$$
\begin{aligned}
DJ_{(\xi,\tau)}(z)\cdot(w_1,\ldots,w_N) &= \omega((\xi,\tau)_P(z),(w_1,\ldots,w_N)) \\
&= -\sum_{\alpha=1}^{N}\Gamma_\alpha \mathrm{Im}\,[(i\xi z_\alpha+\tau)\bar{w}_\alpha] \qquad (32)\\
&= -(\xi,\tau)\cdot\sum_{\alpha=1}^{N}\Gamma_\alpha(z_\alpha\cdot w_\alpha, i w_\alpha), \qquad (33)
\end{aligned}
$$

for all z_j, $w_j \in \mathbb{C}$ shows that

$$
DJ(z)\cdot(w_1,\ldots,w_N) = -\sum_{\alpha=1}^{N}\Gamma_\alpha(z_\alpha\cdot w_\alpha, i w_\alpha). \qquad (34)
$$

The expression (31) for J follows directly from (34). For convenience, let J_1: $\mathbb{C}^N \to \mathbb{R}$ denote the first (rotational) component of the momentum map and J_2: $\mathbb{C}^N \to \mathbb{C}$ denote the second (translational) component.

If a relative equilibrium z were a local extremum of H_ξ, then it would clearly be Lyapunov stable. However, due to the symmetries of the system, nontrivial relative equilibria cannot be strict extremals. The function H_ξ is preserved by the dynamics. Thus we can only hope to obtain a strict minimum if we identify all of the points along the orbit of the relative equilibrium. The vortex dynamics are equivariant with respect to the action of $SO(2)$ on $\mathbb{C}^N$, i.e., the dynamics commute with rotations about the origin; the energy-momentum function H_ξ is invariant under the same action. Thus the equations of motion (1) induce dynamics on the quotient $\mathbb{C}^N/SO(2)$ and H_ξ induces a function on $\mathbb{C}^N/SO(2)$ that is preserved by these dynamics. If the equivalence class of a relative equilibrium is a strict extremal of some function on $\mathbb{C}^N/SO(2)$, then a standard Lyapunov stability argument shows that the equivalence class is nonlinearly stable and hence the relative equilibrium is nonlinearly stable modulo rotations about the origin. Typically the energy-momentum function itself does not determine a Lyapunov function on the quotient; it is necessary to add a $SO(2)$ invariant penalty function associated with the momentum constraint.

Nonlinear orbital stability follows directly from *formal stability* for planar point-vortex systems. Formal stability is closely related to a classic Lagrange multiplier test for an extremal, modified to take into account the symmetry of the system. Specifically, a relative equilibrium is said to be formally stable if the restriction of the second variation of the energy-momentum function to the kernel of the linearized momentum map is semidefinite, with kernel equal to the space of infinitesimal group motions preserving the momentum constraint. In the case at hand, the energy-momentum function is H_ξ and the momentum-preserving group motions are rotations about the origin. Under some fairly general conditions, which are satisfied here, formal stability implies nonlinear orbital stability. For the sake of simplicity and completeness, rather than quote the relevant theorems and verify that the necessary technical assumptions hold, we explicitly construct a modification of the energy-momentum function which determines a Lyapunov function on the quotient $\mathbb{C}^N/SO(2)$.

Proposition 3. *A rigidly rotating relative equilibrium of (1) is nonlinearly $SO(2)$-orbitally stable if it is formally stable. Specifically, the equivalence class of the relative equilibrium in $\mathbb{C}^N/SO(2)$ is a Lyapunov stable fixed point of the dynamics on $\mathbb{C}^N/SO(2)$ induced by the dynamics (1) on $\mathbb{C}^N$.*

Proof. The variations of momentum associated with translations and (real) dilations are

$$DJ(z)\cdot(\tau\,t) = \left(\mathrm{Im}\,[\tau\,J_2(z)],\ \sum_{\alpha=1}^{N}\Gamma_\alpha\tau\right) \qquad \text{and} \qquad DJ(z)\cdot z = (2\,J_1(z),\,J_2(z)), \tag{35}$$

for $\tau\in\mathbb{C}$. Equation (35) implies that $\tau\,t\in\ker[DJ(z)]$ if and only if $\sum_{\alpha=1}^{N}\Gamma_\alpha = 0$ and $\tau\cdot(i\,J_2(z)) = 0$, while $z\in\ker[DJ(z)]$ if and only if $J(z)=0$. Equation (35) and the translation invariance of H imply that $J_2(z)=0$ if $DH_\xi(z)=0$. (We assume throughout that ξ is nonzero.) The second variations of the energy-momentum function with respect to translations and dilations are

$$D^2H_\xi(z)(\tau\,t,\tau\,t) = -\xi\,D^2J_1(z)(\tau\,t,\tau\,t) = \xi\,|\tau|^2\sum_{\alpha=1}^{N}\Gamma_\alpha \tag{36}$$

and

$$D^2H_\xi(z)(z,z) = -\xi\,D^2J_1(z)(z,z) = -2\xi\,J_1(z). \tag{37}$$

These expressions equal zero if the associated variations satisfy the linearized momentum constraint. Hence formal stability is possible only for vortex collections satisfying $\sum_{\alpha=1}^{N}\Gamma_\alpha\neq 0$ and configurations with nonzero momentum.

If z is a formally stable relative equilibrium, then the second variation of the energy-momentum function H_ξ is positive semidefinite on $\ker[DJ(z)]$, but it need not be semidefinite on the entire tangent space $T_z\mathbb{C}^N$. The argument given in the previous paragraph shows that if z is formally stable, then we can take $\mathrm{span}_\mathbb{C}\,[t]\oplus\mathrm{span}_\mathbb{R}\,[z]$ as a complement to $\ker[DJ(z)]$. The second variations of H_ξ, J_1, and J_2 all block diagonalize with respect to the decomposition

$$T_z\mathbb{C}^N = \mathrm{span}_\mathbb{C}\,[t]\oplus\mathrm{span}_\mathbb{R}\,[z]\oplus\ker[DJ(z)]. \tag{38}$$

To control perturbations off the momentum level set we add a penalty function which measures the distance from the level set. At the infinitesimal level, the penalty function should be chosen to have a critical point and a semidefinite second variation at z which dominates $D^2H_\xi(z)$ on the subspace $\mathrm{span}_\mathbb{C}\,[t]\oplus\mathrm{span}_\mathbb{R}\,[z]$.

Let z_e be a formally stable critical point of H_ξ with $J_2(z_e)=0$. If $D^2H_\xi(z_e)$ is positive (negative) definite on $\ker[DJ(z)]$, then for any positive (negative) constant ϵ the augmented energy-momentum function

$$A(z) := H_\xi(z) + \left(\epsilon + \frac{\xi}{4J_1(z_e)}\right)(J_1(z)-J_1(z_e))^2 + \left(\epsilon - \frac{\xi}{2\sum_{\alpha=1}^{N}\Gamma_\alpha}\right)|J_2(z)|^2 \tag{39}$$

has positive (negative) semidefinite second variation

$$D^2A(z_e) = D^2H_\xi(z_e) + \left(\epsilon + \frac{\xi}{2J_1(z_e)}\right) DJ_1(z_e) \otimes DJ_1(z_e) + \left(\epsilon - \frac{\xi}{\sum_{\alpha=1}^N \Gamma_\alpha}\right) DJ_2(z_e) \otimes DJ_2(z_e), \tag{40}$$

with kernel $\mathrm{span}_{\mathbb{R}}\,[iz_e]$, at the critical point z_e. The function A is rotation invariant and hence determines a function $\tilde{A}$ on the quotient $\mathbb{C}^N/SO(2)$ with a local extremum at the equivalence class of z_e. The function $\tilde{A}$ acts as a Lyapunov function on $\mathbb{C}^N/SO(2)$; thus the equivalence class of z_e is nonlinearly stable. □

Having established the relationship between formal and nonlinear stability for the rotating polygonal relative equilibria, we turn to the evaluation of the formal stability conditions. If we define the symmetric $N \times N$ complex matrix $\mathcal{M}(z)$ by

$$\mathcal{M}(z)_{\alpha\beta} = \frac{1}{4\pi}\begin{cases} \dfrac{1}{\Gamma_\alpha}\displaystyle\sum_{\ell\neq\alpha} \Gamma_\ell (z_\alpha - z_\ell)^{-2}, & \alpha = \beta \\ -(z_\alpha - z_\beta)^{-2}, & \alpha \neq \beta, \end{cases} \tag{41}$$

then the second variation of the energy-momentum function H_ξ satisfies

$$D^2H_{(\xi,\tau)}(z)(\delta z, \Delta z) = \overline{\Delta z} \cdot \Gamma\mathcal{M}(z)\Gamma\delta z + \xi\Delta z \cdot \Gamma\delta z. \tag{42}$$

Note that $D^2H_{(\xi,\tau)}(z)$ is *not* a complex bilinear form. The representation of the positions of the point vortices as complex numbers is notationally convenient, but somewhat misleading.

As was noted in the proof of Proposition 3, $\sum_{\alpha=1}^N \Gamma_\alpha \neq 0$ and $J_1(z) \neq 0$ are prerequisites of formal stability, and when these conditions are satisfied the space $\mathrm{span}_{\mathbb{C}}\,[t] \oplus \mathrm{span}_{\mathbb{R}}\,[z]$ is a complement to $\ker[DJ(z)]$ in $T_z\mathbb{C}^N$. In this case, the space $\mathcal{S}$ given by

$$\begin{aligned} \mathcal{S} &:= \{\delta z : DJ(z)\cdot\delta z = 0 = DJ(z)\cdot(i\,\delta z)\} \\ &= \left\{\delta z : \Delta z \in \mathrm{span}_{\mathbb{C}}\,[t, z] \;\Rightarrow\; \delta z \cdot \Gamma\Delta z = 0\right\} \end{aligned} \tag{43}$$

is a complement to $\mathrm{span}_{\mathbb{R}}\,[iz]$ in $\ker[DJ(z)]$, since $DJ_1(z)\cdot z = 2\,J_1(z) \neq 0$ implies that $i\,z \notin \mathcal{S}$. Thus if $\sum_{\alpha=1}^N \Gamma_\alpha$ and $J_1(z)$ are nonzero, then a relative equilibrium z is formally stable if and only if the restriction of $D^2H_\xi(z)$ to $\mathcal{S}$ is definite. The space $\mathcal{S}$ consists of 'shape-changing' variations which alter the relative positions of the vortices. The fact that $\mathcal{S}$ is a complex vector space has some immediate implications for formal stability. The restriction of $D^2H_\xi(z)$ to $\mathcal{S}$ is positive (negative) definite if and only

$$\overline{\delta z}\cdot\mathcal{M}(z)\delta z + \xi\,\delta z \cdot \Gamma^{-1}\delta z > (<)\, 0 \tag{44}$$

and

$$-\overline{\delta z}\cdot\mathcal{M}(z)\delta z + \xi\delta z\cdot\Gamma^{-1}\delta z = \overline{(i\delta z)}\cdot\mathcal{M}(z)(i\delta z) + \xi\,(i\delta z)\cdot\Gamma^{-1}(i\delta z) > (<)\, 0, \tag{45}$$

for all δz such that $\Gamma\delta z \in \mathcal{S}$. The conditions (44) and (45) are in turn equivalent to

$$(-)\,\xi\delta z \cdot \Gamma^{-1}\delta z > \left|\overline{\delta z} \cdot \mathcal{M}(z)\delta z\right|, \tag{46}$$

for all $\delta z \in \mathcal{P}_z := (\mathrm{span}_{\mathbb{C}}\,[t, z])^{\perp}$, the metric orthogonal complement to $\mathrm{span}_{\mathbb{C}}\,[t, z]$. Equation (46) implies that the restriction of the diagonal matrix Γ^{-1} to $\mathcal{P}_z$ is definite if the configuration is formally stable. The sign preceding the left-hand side of the inequality is fixed for all variations; the sign is chosen to be positive (negative) if $\Gamma^{-1}|_{\mathcal{P}_z}$ is positive (negative) definite. The following proposition shows that this condition suffices to show that many classes of polygonal configurations fail the formal stability test. Note that formal instability is not known to imply nonlinear instability and is known *not* to imply linear instability in general.

Proposition 4. *Let z be a steadily rotating vortex configuration containing at least two vortex polygons centered at the origin, that is, assume that there are pairs of natural numbers $\{\iota_1, n_1\}, \ldots, \{\iota_p, n_p\}$, $p \geq 2$, and a renumbering of the vortices such that*

$$\Gamma_\ell = \Gamma_{\iota_j} \qquad \text{and} \qquad z_\ell = z_{\iota_j} e^{2\pi i(\ell - \iota_j)/n_j} \qquad \text{if} \qquad \iota_j < \ell < \iota_j + n_j, \tag{47}$$

for $j = 1, \ldots, p$.

If the restriction of Γ^{-1} to $\mathcal{P}_z$ is positive-definite (respectively, negative-definite), then at most one of the vortex strengths Γ_{ι_j}, $j = 1, \ldots, p$, is negative (positive). If Γ^{-1} is definite on $\mathcal{P}_z$ and there exists $k \in \{1, \ldots, p\}$ such that $\Gamma_{\iota_k}\Gamma_{\iota_j}$ is negative for all $j \neq k$, then the following conditions hold:

1. $\left|\Gamma_{\iota_k}\right| > |\Gamma_{\iota_j}|$ *and* $\left|\Gamma_{\iota_k}\right| |z_{\iota_k}|^2 > |\Gamma_{\iota_j}||z_{\iota_j}|^2$ *for all $j \in \{1, \ldots, p\}$, $j \neq k$,*
2. *the number n_k of vortices in the vortex polygon of strength Γ_{ι_k} is prime.*

Proof. If the vortices $z_{\iota_j}, \ldots, z_{\iota_j+n_j-1}$ (respectively, $z_{\iota_k}, \ldots, z_{\iota_k+n_k-1}$) form an n_j-gon (respectively, an n_k-gon), with all vortices forming the polygon being of equal strength Γ_{ι_j} (respectively, Γ_{ι_k}), then the variations δz and Δz given by

$$\delta z_\ell := c_\ell \qquad \text{and} \qquad \Delta z_\ell := \frac{c_\ell}{\bar{z}_\ell}, \qquad \text{where} \quad c_\ell := \begin{cases} \frac{1}{\sqrt{n_j}}, & \iota_j \leq \ell < \iota_j + n_j \\ -\frac{1}{\sqrt{n_k}}, & \iota_k \leq \ell < \iota_k + n_k, \\ 0, & \text{otherwise} \end{cases} \tag{48}$$

are elements of $\mathcal{P}_z$ satisfying $\Delta z \cdot \Gamma^{-1}\delta z = 0$.

Assume that the restriction of Γ^{-1} is positive-definite; the proof in the case of negative-definiteness is completely analogous. The vector δz given in (48) satisfies

$$\frac{1}{\Gamma_{\iota_j}} + \frac{1}{\Gamma_{\iota_k}} = \delta z \cdot \Gamma^{-1}\delta z > 0. \tag{49}$$

Hence the vortex strength of lesser magnitude must be positive. If two of the polygon vortex strengths are of equal magnitude, both must be positive. The vector Δz given in (48) satisfies

$$\frac{1}{\Gamma_{\iota_j}|z_{\iota_j}|^2} + \frac{1}{\Gamma_{\iota_k}|z_{\iota_k}|^2} = \Delta z \cdot \Gamma^{-1}\Delta z > 0. \tag{50}$$

Thus, considering all possible pairings of vortex polygons in turn, we obtain the first set of conditions.

Assume that a pq-gon, where p and q are both integers, has negative vortex strength Γ_{ι_k}. This pq-gon can, for the purposes of the construction given in (48), be viewed as a collection of p q-gons. The variation $\boldsymbol{\delta z}$ or $\boldsymbol{\Delta z}$ determined by any two of these q-gons yields the required contradiction. □

Remark. The formal instability of the Havelock rings, consisting of two rings with vortex strengths of equal magnitude and opposite sign, is a direct consequence of this corollary.

Given a relative equilibrium z, let $\{\delta z_1, \ldots, \delta z_{N-2}\}$ be an orthonormal basis for the complex vector space $\mathcal{P}_Z$. Let G_Z denote the $(N-2) \times (N-2)$ real matrix with jk-th entry

$$g_{jk} := \delta z_j \cdot \boldsymbol{\Gamma}^{-1} \delta z_k. \tag{51}$$

If the matrix G_Z is indefinite, then z is formally unstable. If G_Z is definite, then the test for formal stability proceeds as follows: Let M_Z denote the $(N-2) \times (N-2)$ complex matrix with jk-th entry

$$m_{jk} := \delta z_j^T \mathcal{M}(z) \delta z_k = \sum_{\ell,m=1}^{N} \mathcal{M}(z)_{\ell m} \delta z_{j\ell} \delta z_{km}. \tag{52}$$

The relative equilibrium z is formally stable if the $2(N-2) \times 2(N-2)$ real matrix with the block structure

$$\begin{pmatrix} \xi\, G_Z + \mathrm{Re}\left[M_Z\right] & -\mathrm{Im}\left[M_Z\right] \\ -\mathrm{Im}\left[M_Z\right] & \xi\, G_Z - \mathrm{Re}\left[M_Z\right] \end{pmatrix} \tag{53}$$

is definite.

We briefly consider the linear stability of relative equilibria. Linearizing the Hamiltonian vector field

$$\mathcal{X}_{H_{(\xi,\tau)}}(z) = -i\boldsymbol{\Gamma}^{-1} D H_{(\xi,\tau)}(z) \tag{54}$$

at a relative equilibrium z with generator (ξ, τ), i.e., a critical point of $H_{(\xi,\tau)}$, yields $L_z = -i\boldsymbol{\Gamma}^{-1} D^2 H_{(\xi,\tau)}(z)$. Let $\mathcal{C}$: $\mathbb{C}^N \to \mathbb{C}^N$ denote the (real) linear map associated with complex conjugation, i.e., $\mathcal{C}\, z = \bar{z}$. Then

$$L_z = -i\,\boldsymbol{\Gamma}^{-1} D^2 H_{(\xi,\tau)}(z) = -i\,\boldsymbol{\Gamma}^{-1}(\mathcal{C}\boldsymbol{\Gamma}\mathcal{M}(z)\boldsymbol{\Gamma} + \xi\boldsymbol{\Gamma}) = -i(\mathcal{C}\mathcal{M}(z)\boldsymbol{\Gamma} + \xi\mathbf{1}), \tag{55}$$

since $\boldsymbol{\Gamma}$ is real and hence commutes with $\mathcal{C}$. Thus once we have computed the matrices G_Z and M_Z appearing in the formal stability analysis of a rotating relative equilibrium, the linearized equations of motion can easily be obtained.

We shall see below that for some simple polygonal configurations the linear and nonlinear stability analyses agree. However, there exist mechanical systems for which this is not the case. 'Gyroscopic stabilization' can cause configurations to be linearly stable which are not local extrema of the energy-momentum function. (See, e.g., Chetaev [1959].) Failure to be a local extremum does *not* imply that a relative equilibrium is not nonlinearly stable, but there are no established techniques available for demonstrating nonlinear stability in this situation.

3.1. Stability of Vortex Polygons

Although the individual entries of (53) are relatively simple, for large N it can be difficult to derive explicit conditions for the definiteness of (53). The problem is simplified if some block structure of the second variation can be used to reduce the size of the matrices to be tested for definiteness. Even though the general algorithm does not guarantee substantial blocking, the symmetries of the problem result in a significant block structure of the second variation beyond that guaranteed by the Hamiltonian structure and continuous symmetries alone.

For example, if n is even, Proposition 1 implies that positivity of the relative vortex strengths $\gamma_0, \ldots, \gamma_{k-1}$ is a prerequisite of formal stability. Thus when searching for stable relative equilibria, we can immediately eliminate any configurations in which some γ_j is negative. In the case $k = 2$ the equations (11) for the relative vortex strengths γ_j take the specific form

$$\gamma_0 = \frac{\sigma\left(1-\rho_0^2\right)}{1-\rho_0/\rho_1} \quad \text{and} \quad \gamma_1 = \frac{\sigma\left(1-\rho_1^2\right)}{1-\rho_1/\rho_0}. \tag{56}$$

As discussed in Section 2, ρ_0^2, ρ_1^2, and ρ_0/ρ_1 are all real. Equation (56) implies that γ_0 and γ_1 are both positive if and only if $\rho_1/\rho_0\left(1-\rho_0^2\right)\left(1-\rho_1^2\right)$ is negative. Thus if n is even, then the only two n-gons, one $2n$-gon configurations which can be formally stable are those satisfying one of the following three conditions:

1. *Vertex-vertex, parallel.* Both the ζ_{0i} and ζ_{1j} lie on the same line through a vertex of the $2n$-gon, with $\sigma > 0$ and either $|\zeta_{0i}| > 1 > \left|\zeta_{1j}\right|$ or $\left|\zeta_{1j}\right| > 1 > |\zeta_{0i}|$.
2. *Vertex-vertex, perpendicular.* The ζ_{0i} and ζ_{1j} lie on perpendicular lines passing through vertices of the $2n$-gon, with $\sigma(1-|\zeta_{\ell i}|) > 0$, $\ell = 0, 1$.
3. *Vertex-face, perpendicular.* ζ_{0i} and ζ_{1j} lie on perpendicular lines passing through midpoints of the faces of the $2n$-gon, with $\sigma > 0$.

These conditions can be expressed in terms of the variables p and q, $p > 0$ and $1 > q > 0$, used in our numerical studies as follows:

1. m and ϕ even, $\frac{1}{q} > p > q$, $f(p, q, n, 0, 0) > 0$.
2. $m + \phi$ odd and either $p > \frac{1}{q}$ and $f(p, q, n, 0, 1) < 0$ or $q > p$ and $f(p, q, n, 1, 0) > 0$.
3. m and ϕ odd, $f(p, q, n, 1, 1) > 0$.

To determine which, if any, of the relative equilibria in the three classes described above are actually stable, it is necessary to consider in detail the matrix (53). To illustrate a possible approach to this analysis and establish the existence of stable members of this family of equilibria, we consider the smallest possible case: $k = n = 2$, i.e., the case of two pairs and a quartet of vortices. The members of the quartet will be assumed to be located on the coordinate axes centered about the origin, with z on the positive real axis.

For an eight-vortex configuration z oriented as described above, the matrix $\mathcal{M}_Z$ takes the value

$$z^2\mathcal{M}(z) = \begin{pmatrix} \Delta_+ & -\frac{i}{2} & -\frac{1}{4} & \frac{i}{2} & k_0^- & k_0^+ & k_1^- & k_1^+ \\ \cdot & \Delta_- & \frac{i}{2} & \frac{1}{4} & \tilde{k}_0^- & \tilde{k}_0^+ & \tilde{k}_1^- & \tilde{k}_1^+ \\ \cdot & \cdot & \Delta_+ & -\frac{i}{2} & k_0^+ & k_0^- & k_1^+ & k_1^- \\ \cdot & \cdot & \cdot & \Delta_- & \tilde{k}_0^+ & \tilde{k}_0^- & \tilde{k}_1^+ & \tilde{k}_1^- \\ \cdot & \cdot & \cdot & \cdot & \Delta_0 & -(4\rho_0)^{-1} & k_m^- & k_m^+ \\ \cdot & \cdot & \cdot & \cdot & \cdot & \Delta_0 & k_m^+ & k_m^- \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Delta_1 & -(4\rho_1)^{-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Delta_1 \end{pmatrix}, \tag{57}$$

where

$$k_j^\pm := -(1 \pm \zeta_j/z)^{-2}, \qquad \tilde{k}_j^\pm := -(i \pm \zeta_j/z)^{-2}, \qquad k_m^\pm := -z^2(\zeta_0 \pm \zeta_1)^{-2}, \tag{58}$$

$$\Delta_\pm := \frac{2\,\gamma_0\,(\rho_0 \pm 1)}{(\rho_0 \mp 1)^2} + \frac{2\,\gamma_1\,(\rho_1 \pm 1)}{(\rho_1 \mp 1)^2} \pm \frac{1}{4}, \tag{59}$$

and

$$\Delta_j := \frac{4\left(\rho_j^2 + 1\right)^2}{\gamma_j\left(\rho_j^2 - 1\right)^2} + \frac{2\,\gamma_{j'}\,(\rho_0 + \rho_1)}{\gamma_j\,(\rho_0 - \rho_1)^2} + \frac{1}{4\,\rho_j}. \tag{60}$$

Here $j' := j + 1 \bmod 2$. The relative vortex strengths γ_0 and γ_1 are given by (56).

We now introduce a basis for the space $\mathcal{P}_z$ with respect to which the matrix (53) block diagonalizes into two (real) six-by-six blocks. This basis is constructed from variations of the form (48). Specifically, if we define

$$\begin{aligned} \boldsymbol{v}_{11} &= (1, i, -1, -i, 0, 0, 0, 0), & \boldsymbol{v}_{21} &= (1, -1, 1, -1, 0, 0, 0, 0), \\ \boldsymbol{v}_{12} &= \left(0, 0, 0, 0, \tfrac{1}{\zeta_0}, -\tfrac{1}{\zeta_0}, -\tfrac{1}{\zeta_1}, \tfrac{1}{\zeta_1}\right), & \boldsymbol{v}_{22} &= (0, 0, 0, 0, 1, 1, -1, -1), \\ \boldsymbol{v}_{13} &= (\nu, -\nu i, -\nu, \nu i, -\zeta_0, \zeta_0, -\zeta_1, \zeta_1), & \boldsymbol{v}_{23} &= (1, 1, 1, 1, -1, -1, -1, -1), \end{aligned} \tag{61}$$

where $\nu := \zeta_0^2 + \zeta_1^2$, then the variations $\boldsymbol{\delta z}_{jk} := \boldsymbol{v}_{jk}/\left|\boldsymbol{v}_{jk}\right|$ form an orthonormal basis for $\mathcal{P}_z$ satisfying

$$\boldsymbol{\delta z}_{1j}^T\,\mathcal{M}(z)\,\boldsymbol{\delta z}_{2k} = 0 \qquad \text{and} \qquad \boldsymbol{\delta z}_{1j} \cdot \boldsymbol{\Gamma}^{-1}\boldsymbol{\delta z}_{2k} = 0, \tag{62}$$

for all j and $k \in \{1, 2, 3\}$.

Let G_j and M_j, $j = 1, 2$, denote the three-by-three matrices defined as in (53) with respect to the basis vectors $\boldsymbol{\delta z}_{j1}$, $\boldsymbol{\delta z}_{j2}$, $\boldsymbol{\delta z}_{j3}$. The relative equilibrium z is formally stable if the matrices

$$\begin{pmatrix} \xi\,G_j + \mathrm{Re}\left[M_j\right] & -\mathrm{Im}\left[M_j\right] \\ -\mathrm{Im}\left[M_j\right] & \xi\,G_j - \mathrm{Re}\left[M_j\right] \end{pmatrix}, \tag{63}$$

are positive-definite for $j = 1, 2$. A numerical check shows that while most of the two pair/quartet relative equilibria are unstable, some of the perpendicular vertex-face configurations are formally stable, and hence nonlinearly orbitally stable. In this case,

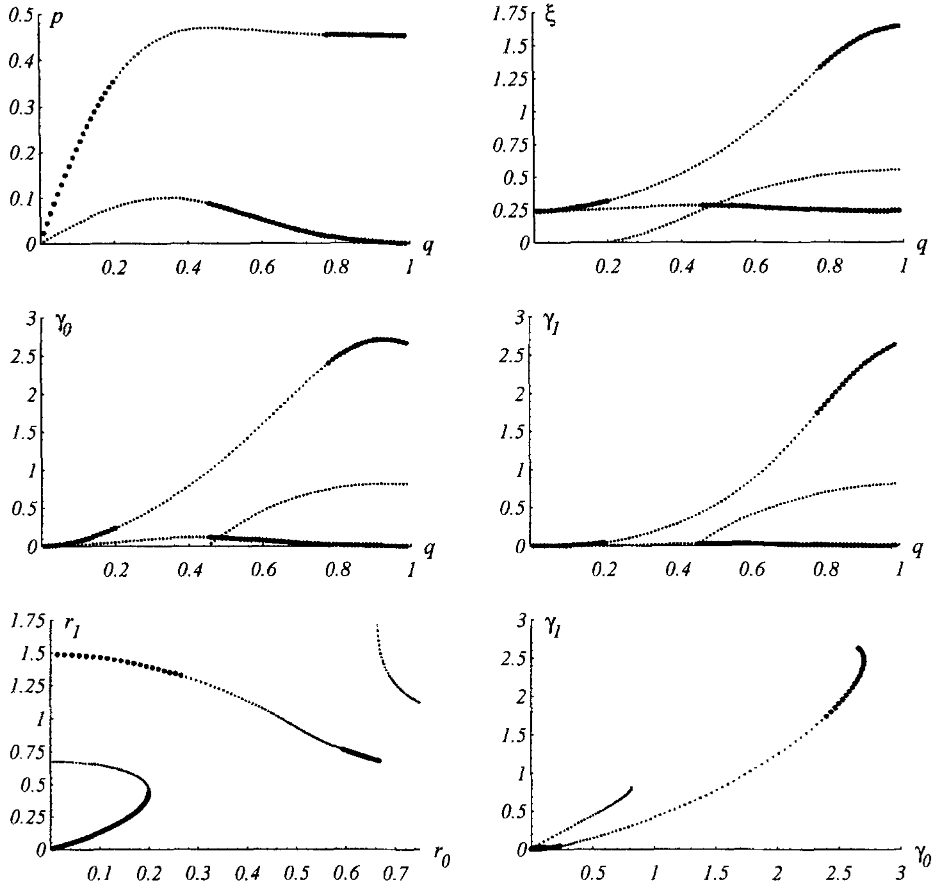

Fig. 5. Stable perpendicular vertex-face equilibria (m and ϕ odd) for $k = n = 2$. Large points indicate stable equilibria, while small points indicate unstable equilibria on the same solution curve.

linear and formal stability precisely coincide. Figure 5 shows the perpendicular vertex-face solutions in terms of the variables (q, p) described in Section 2. The lines of reflectional symmetry of the two pairs are perpendicular and pass through the midpoints of the faces of the square.

Figure 6 illustrates the 'physical' interpretations of the quantities plotted in Figure 5. The stable configuration shown in Figure 6 is determined by the parameter values $k = n = 2$, $m = 1$, $\phi_0 = 0$, $\phi_1 = 1$, and $q = .780$; $p = .454$ is one of the solutions of the equation

$$f(p, q, n, m, \phi_1 - \phi_0) = f(p, 1/q, n, m, \phi_1 - \phi_0),$$

where the function f is given by (30). The relative radii are

$$\frac{|\zeta_0|}{|z|} = r_0 = \sqrt{pq} = .595 \qquad \text{and} \qquad \frac{|\zeta_1|}{|z|} = r_1 = \sqrt{\frac{p}{q}} = .763.$$

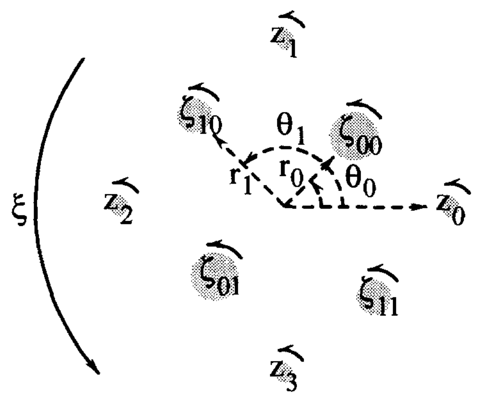

Fig. 6. Stable vertex-face equilibrium configuration for $k = n = 2$, m and ϕ odd, $q = .78$, and $p = .454$.

The relative angles are $\theta_0 = \frac{2\pi}{kn}(k\,\phi_0 + m) = \frac{\pi}{4}$ and $\theta_1 = \frac{2\pi}{kn}(k\,\phi_1 + m) = \frac{3\pi}{4}$. The sizes of the shaded dots reflect the relative vortex strengths $\gamma_0 = 2.402$ and $\gamma_1 = 1.738$ given by (11), while the relative rotation rate

$$\frac{\xi\,|z|^2}{\Gamma} = \frac{n}{2\pi}\left(f(p, q, n, m, \phi_1 - \phi_0) - \frac{1}{2n} + \frac{k}{2}\right) = 16.732$$

is determined from (19) and (30).

Appendix: Technical Lemma and Data Plots

The numerical data for the cases $n = 2$, 3, and 4 are presented in Figures 7–12.

Lemma 1. 1. *Given complex numbers $x_1, \ldots, x_N$, the matrix*

$$\begin{pmatrix} 1 - x_1 & 1 - x_2 & \cdots & 1 - x_N \\ 1 - x_1^2 & 1 - x_2^2 & \cdots & 1 - x_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 - x_1^N & 1 - x_2^N & \cdots & 1 - x_N^N \end{pmatrix} \tag{64}$$

is invertible if and only if $1 \neq x_j \neq x_\ell$ for all $j, \ell = 1, \ldots, N$, $\ell \neq j$.

2. *Let $\mathcal{V}(x_1, \ldots, x_N)$ denote the Vandermonde matrix*

$$\mathcal{V}(x_1, \ldots, x_N) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{pmatrix}, \tag{65}$$

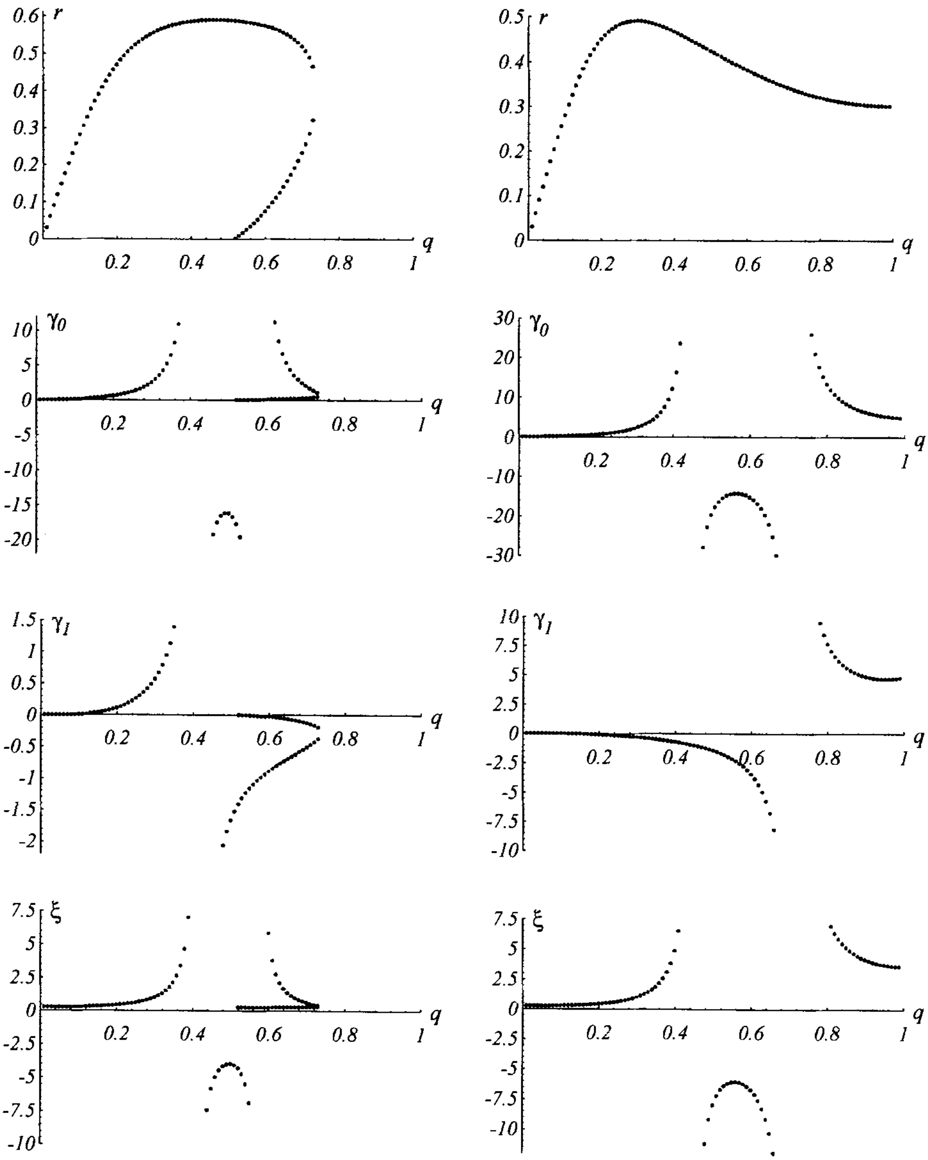

Fig. 7. Data for $k = n = 2$, m even (vertex-vertex). Left: ϕ even; right ϕ odd.

determined by the complex numbers $x_1, \ldots, x_N$. *Then the equation* $\mathcal{V}(x_1, \ldots, x_N)\mathbf{y} = (1, 0, \ldots, 0)$ *has a unique solution* $\mathbf{y}$ *with* j*-th component*

$$y_j := \frac{1}{\prod_{\ell=1,\ell\neq j}^{N}(1 - x_j/x_\ell)}. \tag{66}$$

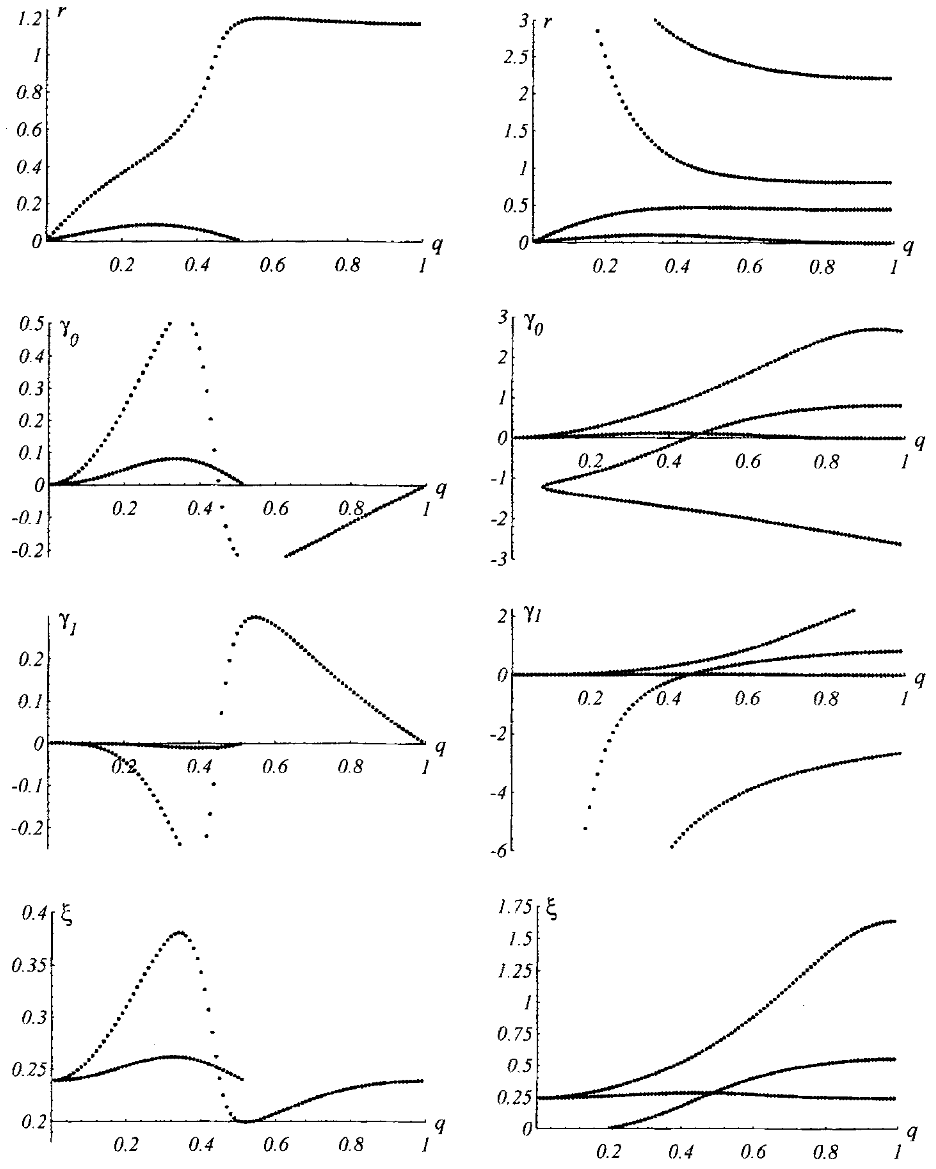

Fig. 8. Data for $k = n = 2$, m odd (vertex-face). Left: ϕ even; right ϕ odd.

Proof. For each integer m, $1 \le m \le N$, define the matrix

$$M_m := \begin{pmatrix} 1 - x_1 & 1 - x_2 & \cdots & 1 - x_m \\ 1 - x_1^2 & 1 - x_2^2 & \cdots & 1 - x_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ 1 - x_1^m & 1 - x_2^m & \cdots & 1 - x_m^m \end{pmatrix}. \tag{67}$$

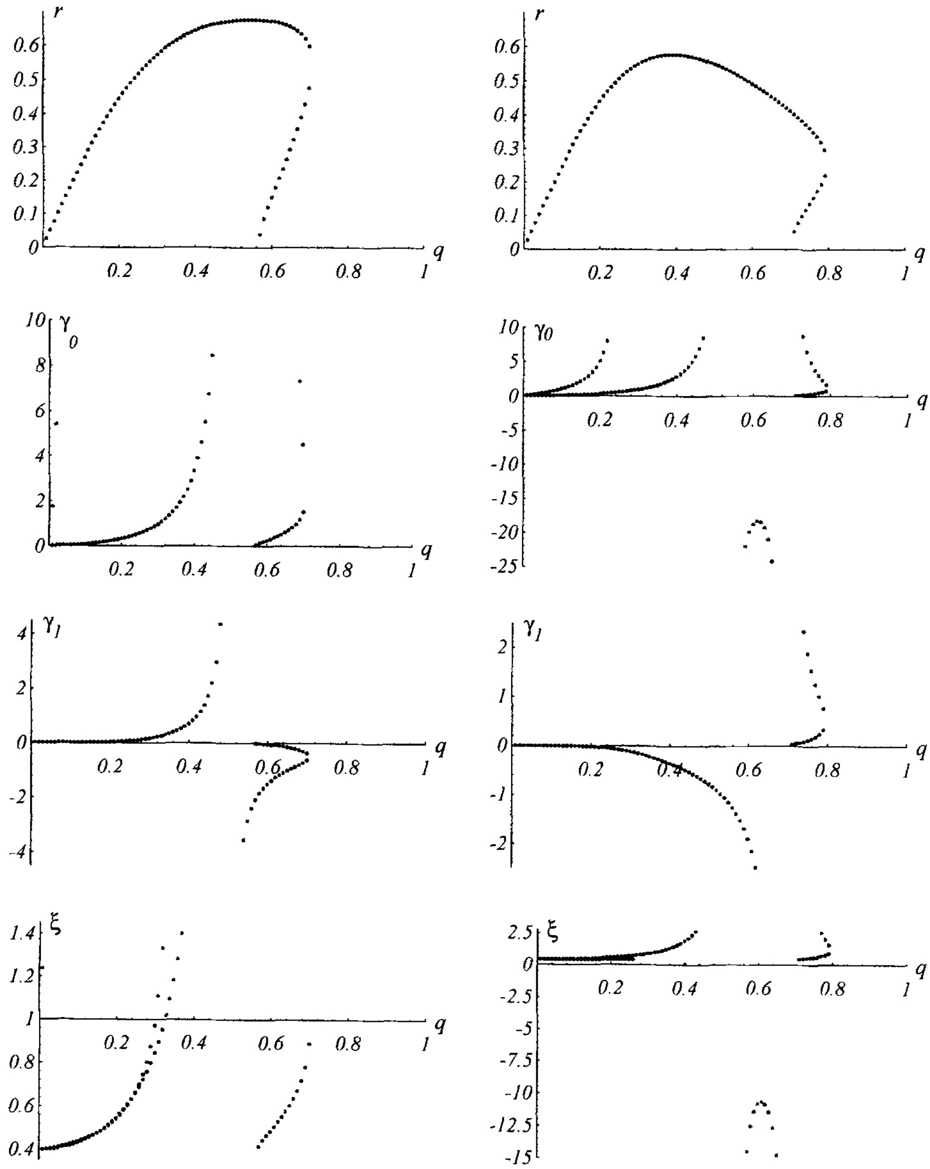

Fig. 9. Data for $k = 2$, $n = 3$, m even (vertex-vertex). Left: ϕ even; right ϕ odd.

For any index j, $1 \leq j \leq m$, we can view the determinant of M_m as an m-th degree polynomial P_{mj} in x_j, with coefficients depending on $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m$. The polynomial P_{mj} has the form

$$P_{mj}(x) = c_{mj}\,(1 - x) \prod_{\ell=1,\ell\neq j}^{m} (x - x_\ell) \tag{68}$$

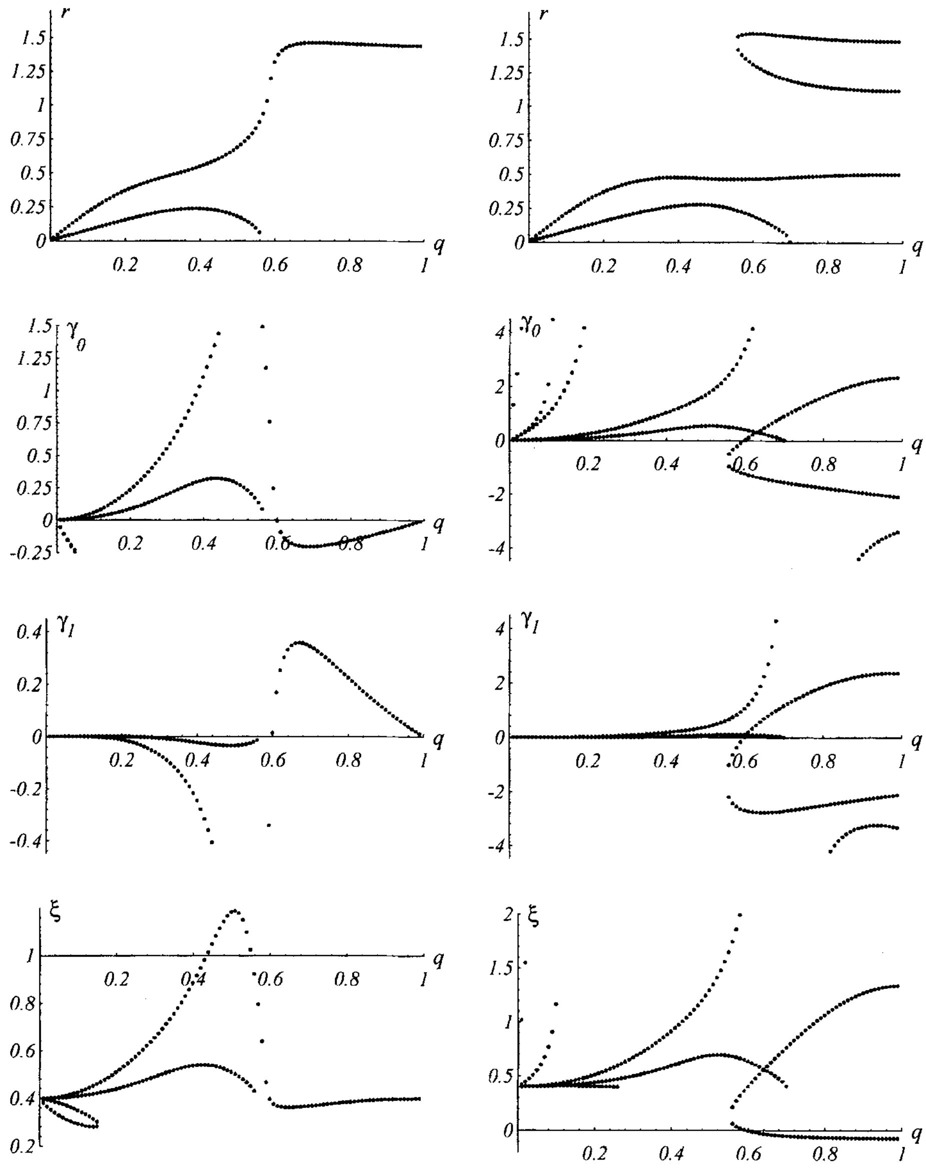

Fig. 10. Data for $k = 2$, $n = 3$, m odd (vertex-face). Left: ϕ even; right ϕ odd.

for some constant c_{mj} depending on $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m$. Since $\det M_m = P_{mj}(x_j)$ for any j, we see that

$$P_{mj}(x_j) = \det M_m = C_m \prod_{j=1}^{m} \left((1 - x_j) \prod_{\ell=1, \ell \neq j}^{m} (x_j - x_\ell) \right) \tag{69}$$

for some constant C_m and any j, $1 \leq j \leq m$.

Clearly $C_1 = 1$; we now show that $C_m = 1$ for all m. Subtracting the first row of M_m from the other rows, and setting $x_m = 0$, we see that

$$\begin{aligned} P_{mm}(0) &= \det \begin{pmatrix} 1-x_1 & 1-x_2 & \cdots & 1-x_{m-1} & 1 \\ x_1(1-x_1) & x_2(1-x_2) & \cdots & x_{m-1}(1-x_{m-1}) & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_1\left(1-x_1^{m-1}\right) & x_2\left(1-x_2^{m-1}\right) & \cdots & x_{m-1}\left(1-x_{m-1}^{m-1}\right) & 0 \end{pmatrix} \\ &= \det M_{m-1} \prod_{j=1}^{m-1} x_j. \end{aligned} \tag{70}$$

On the other hand, (69) implies that

$$P_{mm}(0) = C_m \prod_{j=1}^{m-1} \left(x_j(1-x_j) \prod_{\ell=1,\ell\neq j}^{m-1} (x_j - x_\ell) \right) = \frac{C_m}{C_{m-1}} \det M_{m-1} \prod_{j=1}^{m-1} x_j. \tag{71}$$

Thus $C_m = C_{m-1} = \cdots = C_1 = 1$. It follows from (69) that if $1 \neq x_j \neq x_\ell$ for all $j, \ell = 1, \ldots, m$, $\ell \neq j$, then $\det M_m \neq 0$.

We use Cramer's rule to solve the equation $\mathcal{V}(x_1, \ldots, x_N)\boldsymbol{y} = (1, 0, \ldots, 0)$. A Vandermonde matrix $\mathcal{V}(u_1, \ldots, u_m)$ has determinant

$$\det \mathcal{V}(u_1, \ldots, u_m) = \prod_{j,\ell=1,j\neq\ell}^{m} (u_j - u_\ell). \tag{72}$$

The matrix obtained by replacing the j-th column of $\mathcal{V}(u_1, \ldots, u_m)$ by $(1, 0, \ldots, 0)$ has determinant

$$\begin{aligned} &\det \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{j-1} & 0 & x_{j+1} & \cdots & x_N \\ x_1^2 & x_2^2 & \cdots & x_{j-1}^2 & 0 & x_{j+1}^2 & \cdots & x_N^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_{j-1}^{N-1} & 0 & x_{j+1}^{N-1} & \cdots & x_N^{N-1} \end{pmatrix} \\ &= (-1)^{(j-1)} \det\left(\operatorname{diag}\left[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N\right] \mathcal{V}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)\right) \\ &= (-1)^{(j-1)} \prod_{\ell=1,\ell\neq j}^{N} x_\ell \prod_{\ell=1,\ell'<\ell}^{N} (x_\ell - x'_\ell). \end{aligned} \tag{73}$$

Thus Cramer's rule states that the solution $\boldsymbol{y}$ of the equation $\mathcal{V}(x_1, \ldots, x_N)\boldsymbol{y} = (1, 0, \ldots, 0)$ has j-th component

$$y_j = \frac{(-1)^{(j-1)} \prod_{\ell=1,\ell\neq j}^{N} x_\ell \prod_{\ell=1,\ell'<\ell,\ell\neq j\neq\ell'}^{N} (x_\ell - x'_\ell)}{\prod_{\ell=1,\ell'<\ell}^{N} (x_\ell - x'_\ell)}$$

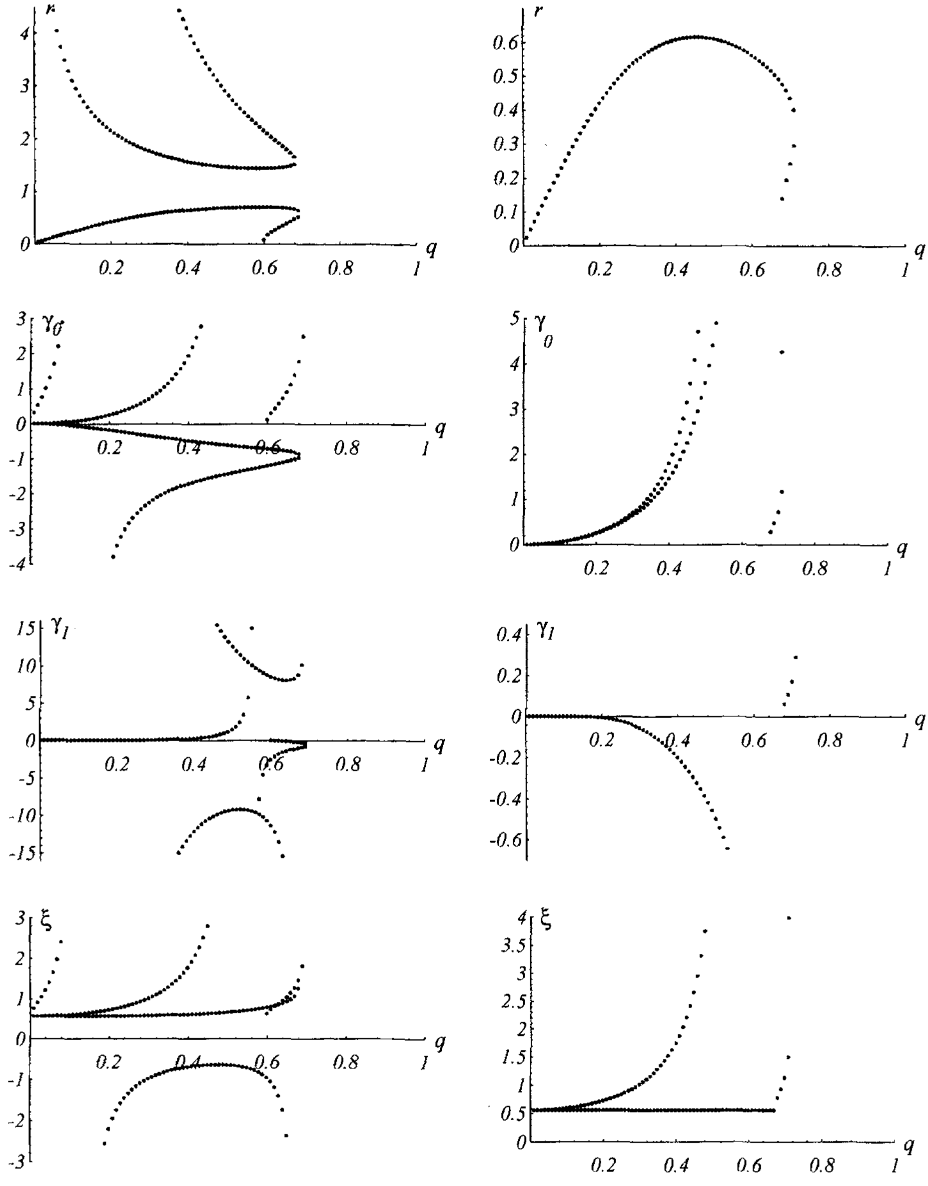

Fig. 11. Data for $k = 2$, $n = 4$, m even (vertex-vertex). Left: ϕ even; right ϕ odd.

$$= \frac{(-1)^{(j-1)} \prod_{\ell=1,\ell\neq j}^{N} x_\ell}{\prod_{\ell=1}^{j-1}(x_j - x_\ell)\prod_{\ell=j+1}^{N}(x_\ell - x_j)}$$

$$= \prod_{\ell=1,\ell\neq j}^{N} \frac{x_\ell}{x_\ell - x_j}. \tag{74}$$

□

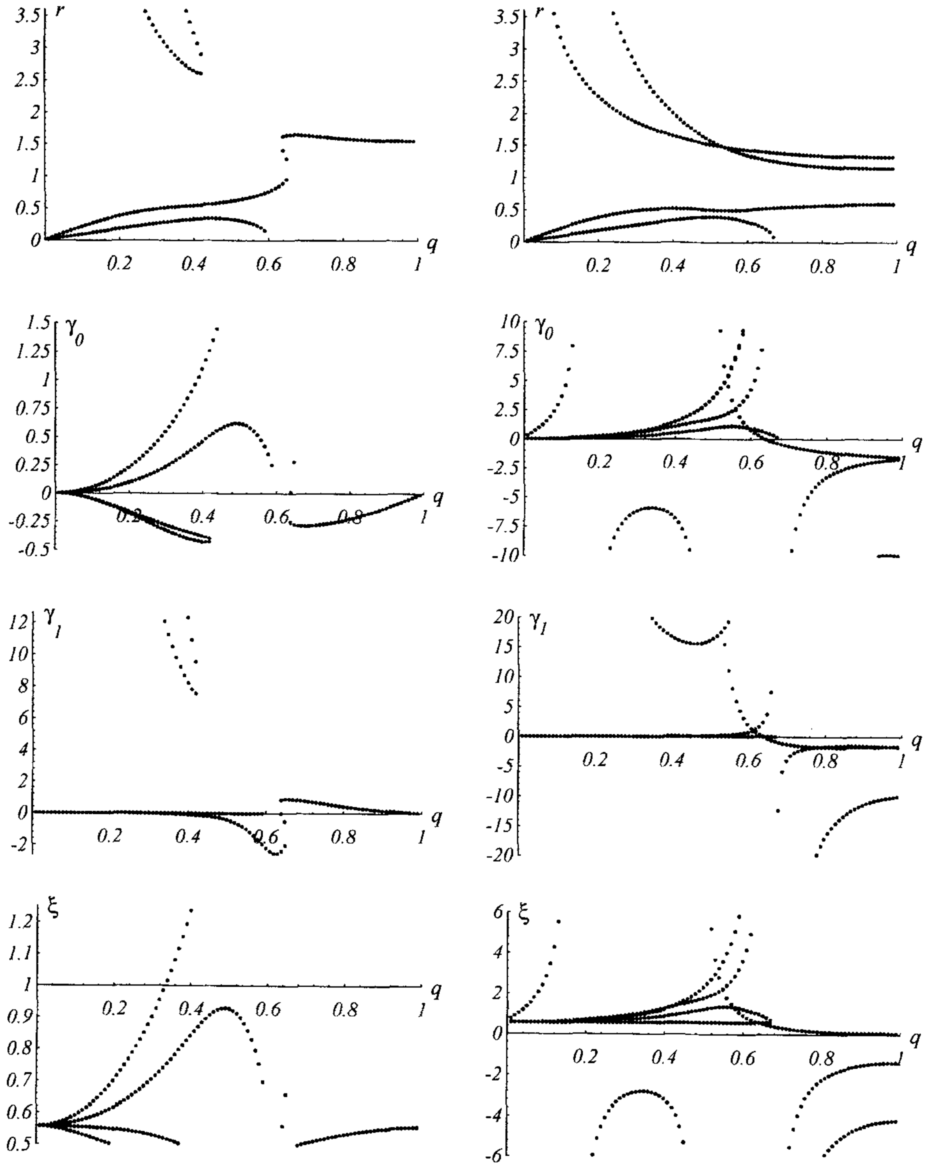

Fig. 12. Data for $k = 2$, $n = 4$, m odd (vertex-face). Left: ϕ even; right ϕ odd.

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